

SCATTERING FOR RADIAL, SEMI-LINEAR, SUPER-CRITICAL WAVE EQUATIONS WITH BOUNDED CRITICAL NORM

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ABSTRACT. In this paper we study the *focusing* cubic wave equation in $1 + 5$ dimensions with radial initial data as well as the one-equivariant wave maps equation in $1 + 3$ dimensions with the model target manifolds \mathbb{S}^3 and \mathbb{H}^3 . In both cases the scaling for the equation leaves the $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ -norm of the solution invariant, which means that the equation is *super-critical* with respect to the conserved energy. Here we prove a conditional scattering result: If the critical norm of the solution stays bounded on its maximal time of existence, then the solution is global in time and scatters to free waves as $t \rightarrow \pm\infty$. The methods in this paper also apply to all supercritical power-type nonlinearities for both the focusing and defocusing radial semi-linear equation in $1 + 5$ dimensions, yielding analogous results.

1. INTRODUCTION

In this paper we study three super-critical semi-linear wave equations, namely the focusing cubic wave equation in \mathbb{R}^{1+5} with radial initial data and the one-equivariant wave maps equations from $\mathbb{R}^{1+3} \rightarrow \mathbb{S}^3$ and from $\mathbb{R}^{1+3} \rightarrow \mathbb{H}^3$. Under certain conditions the former equation serves as a good model for the first of the latter two, which have nonlinearities that arise naturally from the geometry of the target manifold.

1.1. The cubic wave equation in \mathbb{R}^{1+5} . Consider first the Cauchy problem for the focusing cubic semi-linear wave equation in \mathbb{R}^{1+5} ,

$$\begin{aligned} u_{tt} - \Delta u - u^3 &= 0, \\ \vec{u}(0) &= (u_0, u_1), \end{aligned} \tag{1.1}$$

restricted to the radial setting. The conserved energy for solutions,

$$\vec{u}(t) := (u(t), u_t(t)),$$

to (1.1) is given by

$$E(\vec{u}(t)) := \int_{\mathbb{R}^5} \left[\frac{1}{2} (|u_t(t)|^2 + |\nabla u(t)|^2) - \frac{1}{4} |u(t)|^4 \right] dx = \text{constant}.$$

As we will only be considering radial solutions to (1.1), we slightly abuse notation by often writing $u(t, x) = u(t, r)$ where here (r, ω) with $r = |x|$, $x = r\omega$, $\omega \in \mathbb{S}^4$ are polar coordinates on \mathbb{R}^5 . In this setting we can rewrite the equation (1.1) as

$$\begin{aligned} u_{tt} - u_{rr} - \frac{4}{r} u_r - u^3 &= 0, \\ \vec{u}(0) &= (u_0, u_1), \end{aligned} \tag{1.2}$$

Support of the National Science Foundation, DMS-1103914 for the first author, and DMS-1302782 for the second author, is gratefully acknowledged.

and the conserved energy (up to a constant multiple) by

$$E(\vec{u}(t)) := \int_0^\infty \left[\frac{1}{2}(u_t^2(t, r) + u_r^2(t, r)) - \frac{1}{4}u^4(t, r) \right] r^4 dr. \quad (1.3)$$

The Cauchy problem (1.2) is invariant under the scaling

$$\vec{u}(t, r) \mapsto \vec{u}_\lambda(t, r) := (\lambda^{-1}u(t/\lambda, r/\lambda), \lambda^{-2}u_t(t/\lambda, r/\lambda)). \quad (1.4)$$

One can also check that this scaling leaves unchanged the $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ -norm of the initial data. It is for this reason that (1.2) is called *energy-supercritical*.

The standard argument based on Strichartz estimates shows that (1.2) is locally well-posed in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$. This means that for all initial data $\vec{u}(0) = (u_0, u_1) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$, there is a unique solution, $\vec{u}(t)$, defined on a maximal interval of existence $I_{\max} = I_{\max}(\vec{u})$ with $\vec{u} \in C^0(I_{\max}; \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5))$. Moreover, for every compact time interval $J \subset I_{\max}$ we have

$$u \in S(J) := L_t^2(J; L_x^{10}(\mathbb{R}^5)). \quad (1.5)$$

The Strichartz norm $S(J)$ determines a criterion for both scattering and finite time blow up, see Proposition 2.4. In particular, one can show that if the initial data $\vec{u}(0)$ is sufficiently small in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$, then the corresponding solution $\vec{u}(t)$ has finite $S(\mathbb{R})$ -norm and hence scatters to free waves as $t \rightarrow \pm\infty$.

The theory for solutions to (1.2) with initial data that is small in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ is thus well understood – all solutions are global in time and scatter to free waves as $t \rightarrow \pm\infty$. However, much less is known regarding the *asymptotic dynamics* of solutions once one leaves the perturbative regime.

There are solutions to the focusing problem that blow-up in finite time. For example,

$$\varphi_T(t, x) = \frac{\sqrt{2}}{T - t}$$

solves the ODE, $\varphi_{tt} = \varphi^3$. By the finite speed of propagation, one can construct from φ_T a compactly supported (in space) self-similar blow-up solution to (1.2), $\vec{u}_T(t)$, with blow-up time $t = T$. However, such a self-similar solution must have

$$\lim_{t \rightarrow T} \|\vec{u}_T(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} = \infty.$$

Such behavior is typically referred to as type-I blow-up. On the other hand, type-II solutions, $\vec{u}(t)$, are those whose critical norm remains bounded on their maximal interval of existence, I_{\max} , i.e.,

$$\sup_{t \in I_{\max}} \|\vec{u}(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} < \infty. \quad (1.6)$$

In this paper we restrict our attention to type-II solutions, i.e., those which satisfy the bound (1.6). We prove that if a solution $\vec{u}(t)$ to (1.2) satisfies (1.6), then it must exist globally in time and scatter to free waves in both time directions. We establish the following result.

Theorem 1.1. *Let $\vec{u}(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ be a radial solution to (1.2) defined on its maximal interval of existence $I_{\max} = (T_-, T_+)$. Suppose in addition that*

$$\sup_{t \in I_{\max}} \|\vec{u}(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} < \infty. \quad (1.7)$$

Then, $I_{\max} = \mathbb{R}$, that is, $\vec{u}(t)$ is defined globally in time. Moreover,

$$\|u\|_{S(\mathbb{R})} < \infty, \quad (1.8)$$

which means that $\vec{u}(t)$ scatters to free waves as $t \rightarrow \pm\infty$, i.e., there exist radial solutions $\vec{u}_L^\pm(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ to the free wave equation, $\square u_L = 0$, so that

$$\|\vec{u}(t) - \vec{u}_L^\pm(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (1.9)$$

Remark 1. Theorem 1.1 is a conditional result. Other than the requirement that the initial data be small in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$, there is no known general criterion to determine when (1.7) is satisfied by the evolution. We remark that that this type of result is analogous to the work by Duyckaerts, Kenig, and Merle in 3 dimensions in [20] and also bears similarity to the $L^{3,\infty}$ result of Escauriaza, Sřegin, and řverak for the Navier-Stokes equation, see [22].

Remark 2. The proof of Theorem 1.1 readily generalizes to all supercritical powers $p > \frac{7}{3}$ in dimension $d = 5$ for both the focusing and defocusing equations with radial initial data. As we demonstrate in the next subsection where we consider the case of one-equivariant wave maps, the techniques in this paper are very flexible with regards to particular algebraic structure of the nonlinearity. For power-type nonlinearities we have chosen to present the details for only the focusing cubic equation to keep the exposition as simple as possible. Another reason for choosing to study the *focusing* cubic equation as opposed to other power-type nonlinearities is this equation requires several new techniques which fall outside the scope of what has previously been developed in the literature, such as [20], which is the first conditional result for a focusing supercritical equation, and [29, 30, 34, 36, 5], which address defocusing supercritical waves.

1.2. One-Equivariant Wave Maps. Next we consider one-equivariant wave maps in $1+3$ dimensions taking values in 3-dimensional rotationally symmetric manifolds \mathcal{M} . Let $(r, \theta) \in \mathbb{R}^* \times \mathbb{S}^2$ be polar coordinates on \mathbb{R}^3 and let (ψ, ω) be geodesic polar coordinates on \mathcal{M} , where the metric takes the form

$$ds^2 = dr^2 + g^2(\psi)d\omega^2$$

and $d\omega^2$ denotes the round metric on \mathbb{S}^2 . Maps $U : \mathbb{R}^{1+3} \rightarrow \mathcal{M}$ can then be written in the form $U(t, r, \theta) = (\psi(t, r, \theta), \omega(t, r, \theta))$. In the usual one-equivariant (or co-rotational) reduction, one makes the ansatz

$$U(t, r, \theta) = (\psi(t, r), \theta) \quad (1.10)$$

and the wave maps system reduces to a Cauchy problem for the coordinate function ψ , viz.,

$$\begin{aligned} \psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{f(\psi)}{r^2} &= 0 \\ \vec{\psi}(0) &= (\psi_0, \psi_1) \end{aligned} \quad (1.11)$$

where $f(\psi) := g(\psi)g'(\psi)$. The conserved energy is given by

$$\mathcal{E}(\vec{\psi})(t) := \frac{1}{2} \int_0^\infty \left[\psi_t^2 + \psi_r^2 + \frac{2g^2(\psi)}{r^2} \right] r^2 dr = \text{constant}. \quad (1.12)$$

We also note the scaling invariance,

$$\psi(t, r) \mapsto \psi_\lambda(t, r) := \psi(t/\lambda, r/\lambda) \quad (1.13)$$

Note that it is energetically favorable for a solution to concentrate to a point by the rescaling above and sending $\lambda \rightarrow 0$ as we have

$$\mathcal{E}(\vec{\psi}_\lambda) = \lambda \mathcal{E}(\vec{\psi}) \rightarrow 0 \quad \text{as } \lambda \rightarrow 0 \quad (1.14)$$

This means that 1+3 dimensional wave maps are *super-critical* with respect to the conserved energy. It is well known that 3d wave maps into positively curved targets such as the 3-sphere, $\mathcal{M} = \mathbb{S}^3$, can blow up in finite time in a self-similar fashion. This was proved by Shatah in [50] and an explicit example was given by Turok, Spergel [60] with

$$\psi(t, r) = 2 \arctan(r/t). \quad (1.15)$$

For negatively curved targets such as 3d-hyperbolic space, $\mathcal{M} = \mathbb{H}^3$, it is not known whether singularities can develop in finite time.

Wave maps arise as a model in particle physics, particularly with dimension $d = 3$, and in this setting are referred to as nonlinear σ -models. Here for simplicity we will consider two model targets, namely $\mathcal{M} = \mathbb{S}^3$ and $\mathcal{M} = \mathbb{H}^3$.

1.2.1. *Wave maps into \mathbb{S}^3 .* The case of the target $\mathcal{M} = \mathbb{S}^3$ has traditionally been of interest due to the possibility of solutions with nontrivial topology. In our equivariant formulation with the \mathbb{S}^3 target, we have $g(\psi) = \sin \psi$ and the equation and conserved energy become

$$\begin{aligned} \psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} &= 0, \quad \vec{\psi}(0) = (\psi_0, \psi_1) \\ \mathcal{E}(\vec{\psi})(t) &:= \frac{1}{2} \int_0^\infty \left[\psi_t^2 + \psi_r^2 + \frac{2\sin^2(\psi)}{r^2} \right] r^2 dr \end{aligned} \quad (1.16)$$

From the above it is clear that for initial data $\vec{\psi}(0) = (\psi_0, \psi_1)$ to have finite energy one requires that $\lim_{r \rightarrow \infty} \psi_0(r) = n\pi$ for some $n \in \mathbb{N}$, and by a continuity argument, this endpoint is fixed by the evolution. The energy allows for more flexible behavior of $\psi_0(r)$ at $r = 0$ in contrast to the case of 2d wave maps into the 2-sphere, which have topological degree which is fixed by the evolution. Here we see that simply requiring that the solution has finite energy allows for any finite limit $\lim_{r \rightarrow 0} \psi(t, r) = \alpha(t) \in \mathbb{R}$ and this limit can possibly change with the evolution. Here, our techniques force us to ignore this subtlety as will impose the priori assumption that for all $t \in I_{\max}(\vec{\psi})$ we have

$$\vec{u}(t, r) := (u(t, r), u_t(t, r)) := \left(\frac{\psi(t, r)}{r}, \frac{\psi_t(t, r)}{r} \right) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5) \quad (1.17)$$

With this assumption we can recast the Cauchy problem (1.16) in terms of $\vec{u}(t)$, which solves

$$\begin{aligned} u_{tt} - u_{rr} - \frac{4}{r}u_r &= \frac{2ru - \sin(2ru)}{r^3} := F_{\mathbb{S}^3}(r, u) \\ \vec{u}(0) &= (u_0, u_1) \end{aligned} \quad (1.18)$$

By Sobolev embedding we must then have $u(t) \in L^5(\mathbb{R}^5)$ for all $t \in I_{\max}$, which in turn implies that

$$\int_0^\infty \frac{\psi^5(t, r)}{r} dr < \infty.$$

Hence we must have $\psi(t, 0) = 0$ and $\psi(t, \infty) = 0$ for all $t \in I_{\max}$ once we make the a priori assumption (1.17). Note also that the nonlinearity satisfies

$$\begin{aligned} F_{\mathbb{S}^3}(r, u) &= u^3 Z_{\mathbb{S}^3}(ru) \\ |F_{\mathbb{S}^3}(r, u)| &\lesssim |u|^3 \end{aligned} \quad (1.19)$$

where $Z_{\mathbb{S}^3}(\rho) = 8 \frac{\rho - \sin \rho}{\rho^3}$ is a smooth bounded, *nonnegative* function.

Thus, in the $5d$ formulation (1.18), the topologically trivial equivariant wave maps problem into \mathbb{S}^3 bears many similarities to the *focusing* cubic equation (1.2). We establish a conditional result, which is completely analogous to Theorem 1.1. Before stating our main theorem regarding equivariant wave maps, we first introduce the case the of the \mathbb{H}^3 target.

1.2.2. Wave maps into \mathbb{H}^3 . The case of the negatively curved target $\mathcal{M} = \mathbb{H}^3$ bears many similarities to the defocusing radial cubic wave equation in \mathbb{R}^{1+3} after the reduction performed below. In the equivariant formulation with the \mathbb{H}^3 target, we have $g(\psi) = \sinh \psi$ and the equation and conserved energy become

$$\begin{aligned} \psi_{tt} - \psi_{rr} - \frac{2}{r} \psi_r + \frac{\sinh(2\psi)}{r^2} &= 0, \quad \vec{\psi}(0) = (\psi_0, \psi_1) \\ \mathcal{E}(\vec{\psi})(t) &:= \frac{1}{2} \int_0^\infty \left[\psi_t^2 + \psi_r^2 + \frac{2 \sinh^2(\psi)}{r^2} \right] r^2 dr \end{aligned} \quad (1.20)$$

From the above it is clear that for initial data $\vec{\psi}(0) = (\psi_0, \psi_1)$ to have finite energy one requires that $\lim_{r \rightarrow \infty} \psi_0(r) = 0$ and this endpoint is fixed by energy conservation. As in the case of the \mathbb{S}^3 target, the energy allows for more flexible behavior of $\psi_0(r)$ at $r = 0$. Here we see that simply requiring that the solution has finite energy allows for any finite limit $\lim_{r \rightarrow 0} \psi(t, r) = \alpha(t) \in \mathbb{R}$ and this limit can possibly change with the evolution. Here, again we ignore this subtlety and impose the priori assumption that for all $t \in I_{\max}(\vec{\psi})$ we have

$$\vec{u}(t, r) := (u(t, r), u_t(t, r)) := \left(\frac{\psi(t, r)}{r}, \frac{\psi_t(t, r)}{r} \right) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5) \quad (1.21)$$

With this assumption we can recast the Cauchy problem (1.20) in terms of $\vec{u}(t)$, which solves

$$\begin{aligned} u_{tt} - u_{rr} - \frac{4}{r} u_r &= \frac{2ru - \sinh(2ru)}{r^3} := F_{\mathbb{H}^3}(r, u) \\ \vec{u}(0) &= (u_0, u_1) \end{aligned} \quad (1.22)$$

Assuming (1.21) and by Sobolev embedding we must then have $u(t) \in L^5(\mathbb{R}^5)$ for all $t \in I_{\max}$, which in turn implies that

$$\int_0^\infty \frac{\psi^5(t, r)}{r} dr < \infty.$$

Hence we must have $\psi(t, 0) = 0$ once we make the a priori assumption (1.17). Note also that the nonlinearity satisfies

$$F_{\mathbb{H}^3}(r, u) = u^3 Z_{\mathbb{H}^3}(ru) \quad (1.23)$$

where $Z_{\mathbb{H}^3}(\rho) = 8 \frac{\rho - \sinh \rho}{\rho^3}$ is a smooth *nonpositive* function. If we assume an a priori uniform bound

$$\sup_{t \in I_{\max}} \|\vec{u}(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} \leq C < \infty, \quad (1.24)$$

then we have L^∞ control on $\psi = ru$ by radial Sobolev embedding,

$$|ru(t, r)| \lesssim \sup_{t \in I_{\max}} \|\vec{u}(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} \lesssim C, \quad (1.25)$$

which follows from Lemma 2.2. With the assumption (1.24), we thus have a uniform bound on $Z_{\mathbb{H}^3}(ru)$ and hence

$$|F_{\mathbb{H}^3}(r, u)| \lesssim |u|^3 \quad (1.26)$$

making the analogy with the defocusing cubic equation in \mathbb{R}^{1+5} clear.

Finally, we state our main result for wave maps, which holds for both the \mathbb{S}^3 target, (1.18), and for the \mathbb{H}^3 target, (1.22).

Theorem 1.2. *Let $\vec{u}(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ be a radial solution to either (1.18) or to (1.22) defined on its maximal interval of existence $I_{\max} = (T_-, T_+)$. Suppose in addition that*

$$\sup_{t \in I_{\max}} \|\vec{u}(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} < \infty. \quad (1.27)$$

Then, $I_{\max} = \mathbb{R}$, that is, $\vec{u}(t)$ is defined globally in time. Moreover,

$$\|u\|_{S(\mathbb{R})} < \infty, \quad (1.28)$$

which means that $\vec{u}(t)$ scatters to free waves as $t \rightarrow \pm\infty$, i.e., there exist radial solutions $\vec{u}_L^\pm(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ to the free wave equation, $\square u_L = 0$, so that

$$\|\vec{u}(t) - \vec{u}_L^\pm(t)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)} \longrightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (1.29)$$

1.3. History of the problems. There is very little known about energy super-critical semi-linear wave equations on \mathbb{R}^{1+d} , at least when compared to the vast body of literature devoted to their sub-critical and critical counterparts.

There are several conditional results in the same vein as Theorem 1.1 and Theorem 1.2 in the case of *defocusing* super-critical equations, where Morawetz type identities can be used; see for example [29, 30, 36, 37, 4, 5]. The only such results for *focusing* type equations as considered here are the work of Duyckaerts, Kenig, and Merle, [20], concerning super-critical power-type nonlinearities in dimension $d = 3$, and the work of the second author on a semi-linear Skyrme-type equation in [45].

There is much less know in the way of unconditional results. Recently, in an exciting new direction, Krieger and Schlag, [43], have constructed a family of solutions to the super-critical power type equation in $d = 3$, which are smooth, global-in-time, have *infinite* critical norm, and are stable under small perturbations.

For the focusing NLS and focusing wave equations in high dimensions, $d \geq 11$, there have recently been blow-up constructions based on the bubbling off of a solution to the underlying elliptic equation; see Merle, Raphael, Rodnianski, [47] and Collot, [8] (we note that in the aforementioned works something different is meant by “type II” than how this phrase is used in this paper).

Super-critical equivariant wave maps in $d = 3$ as considered here are called non-linear σ -models in particle physics and have been extensively studied. As we mentioned above, self-similar blow-up was demonstrated by Shatah, [50], and Turok-Spergel in the case of the \mathbb{S}^3 target. Donninger has established stability for such self-similar solutions, [11, 12]; see also the work of Bizon, [2]. Similar results have been established for power-type nonlinearities, see [15] for a stability result, and [3] for a construction of an infinite family of smooth, self-similar solutions. Surprisingly, blow-up can occur even in the case of wave maps into negatively curved targets in high enough dimensions, as shown by Cazenave, Shatah, and Tahvildar-Zadeh, [6]; see the book of Shatah and Stuwe [51] for more.

Equations of the form

$$\square u = \pm |u|^{p-1} u \quad (1.30)$$

for energy critical and sub-critical values of p have been extensively studied. When say, $d = 3$, the energy critical power, $p = 5$, exhibits markedly different phenomena than both the subcritical and supercritical problems. Global existence and scattering for all finite energy data was established by Struwe, [54], for the radial defocusing equation and by Grillakis, [23], in the nonradial, defocusing case.

In the case of the focusing energy critical equation, type-II blow up does occur, as explicitly demonstrated by Krieger, Schlag, and Tataru [44], by way of an energy concentration scenario resulting in the bubbling off of the unique radial ground state solution, W , for the underlying elliptic equation; see also [42, 13, 14].

In [27], Kenig and Merle initiated an extremely effective program of attack for semilinear equations such as (1.30) with the concentration compactness/rigidity method based on the fundamental profile decompositions of Bahouri and Gérard, [1]. There they gave a characterization of possible dynamics for solutions with energy strictly below the threshold energy of the ground state elliptic solution, W . The seminal work of Duyckaerts, Kenig, and Merle [16, 17, 18, 19] gave a classification of possible dynamics for large energies. To be more precise, all type-II radial solutions asymptotically resolve into a sum of rescaled solitons plus a radiation term. Dynamics at the threshold energy of W have been studied by Duyckaerts and Merle [21] and slightly above the threshold energy by Krieger, Nakanishi, and Schlag in [39, 40, 41].

For analogues of Theorem 1.1 and Theorem 1.2 for energy sub-critical equations we refer the reader to [52], and to the recent work of the authors, [10]. We also mention the remarkable works of Merle and Zaag, [48, 49] where it was determined that all subcritical blow-up for the focusing equation occurs at the self-similar rate.

1.4. Outline of the proofs of Theorem 1.1 and Theorem 1.2. The proofs of Theorem 1.1 and Theorem 1.2 proceed via the concentration compactness/ rigidity method developed by Kenig and Merle in [26, 27]. The method is based around an elaborate contradiction argument – if Theorem 1.1 (respectively Theorem 1.2) were false, the linear and nonlinear profile decompositions of Bahouri and Gérard [1] allow for a construction of a minimal non-scattering solution to (1.2), called the critical element. Here minimality refers to the size of the norm in (1.7) (resp. (1.27)). This construction is standard in the field and we give a brief outline in Section 3. The crucial property of the critical element that drives the contradiction argument is that its trajectory is *pre-compact* up to modulation in the space $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$. The

goal is then to prove that this compactness property is too restrictive for a nonzero solution and therefore the critical element does not exist.

For the rigidity argument that rules out the critical element, we roughly follow the strategy implemented in [20] for the focusing super-critical wave equation in 3-dimensions. Given the *exterior energy estimates* for the free radial wave equation proved in [25], the precise strategy in [20] can be adapted to super-critical equations of the form

$$\square u = \pm |u|^{p-1} u \quad (1.31)$$

in higher dimensions, in particular $d = 5$, *but only when* $p > 5$, i.e., for critical regularity levels $\dot{H}^s \times \dot{H}^{s-1}$ with $s \geq 2$. There are several instances where the techniques used in [20] break down for supercritical powers $p < 5$, and in particular for the cubic-type equations considered here. Here we build on the strategy developed in [20], by developing collection of robust new techniques that work for all powers $p > \frac{7}{3}$ as well as for more complicated nonlinearities such as those which arise in the context of equivariant wave maps, as in (1.18) and (1.22).

As in [20] we reduce to two scenarios for the critical element $\vec{u}^*(t)$, namely,

- (1) $\vec{u}^*(t)$ is a self-similar blow-up solution with pre-compact trajectory, see Proposition 3.5.
- (2) $\vec{u}^*(t)$ is pre-compact on its entire interval of existence I up to a modulation parameter $N(t)$ that is bounded away from 0 on I , see Proposition 3.4.

In the first situation, (1), we follow [20] by introducing self-similar coordinates and using compactness to produce a stationary solution to a non-degenerate elliptic equation with zero boundary data. However, a straightforward application of their approach adapted to $5d$ breaks down for the cubic equation and the regularity $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$. We overcome this difficulty by proving, in Section 4, a bound on the $\dot{H}^{\frac{5}{2}} \times \dot{H}^{\frac{3}{2}}$ norm of $\vec{u}^*(t)$, i.e., we show that solutions with pre-compact trajectories are in fact more regular than what is given by scaling; see Proposition 4.1 and Proposition 4.6. This bound on the $\dot{H}^{\frac{5}{2}} \times \dot{H}^{\frac{3}{2}}$ norm of $\vec{u}^*(t)$ then implies (by interpolation) that a self-similar compact blow-up solution is in fact also compact in $\dot{H}^2 \times \dot{H}^1 \cap \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$. With this additional regularity we are able to rule out compact self-similar blow up with a somewhat simpler implementation of the argument in [20]. The crucial gain of regularity in Section 4 is established using the so-called double Duhamel trick which we will describe briefly below. We note that an analogous implementation of the double Duhamel trick was performed by the authors in [10] for the cubic equation in $d = 3$.

There are several major difficulties when trying to rule out a critical element $\vec{u}^*(t)$ as in (2) for supercritical equations as considered here. The first is that $\vec{u}^*(t)$ is constructed in the homogeneous space $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$. Although having a pre-compact trajectory up to modulation in this space is a strong property, there is very little in practice that one can do at this regularity, since useful global quantities such as the conserved energy are at the level of $(\dot{H}^1 \cap \dot{H}^{\frac{5}{4}}) \times L^2$, where here the intersection with $\dot{H}^{\frac{5}{4}}$ arises due to the $L^4(\mathbb{R}^5)$ term in the conserved energy associated to (1.2) and Sobolev embedding. In order to resolve this first issue, we prove in Section 4 that solutions $\vec{u}^*(t)$ as in (2) have more decay than what is given by scaling. In particular, we use the another implementation of the double Duhamel trick to prove that in fact a compact solution as in (2) above must be uniformly

bounded in $\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}$. Interpolating with the critical $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ bound then implies compactness in the space $\dot{H}^1 \times L^2$, see Lemma 6.1. An alternative method for establishing additional spacial decay based on delicate flux-type estimates was used in [29, 30, 20]. However, this method does not seem to work for the cubic type nonlinearities considered here, thus motivating the different approach based on the double-Duhamel trick.

Second, even once the decay issue is resolved, there are no known useful viral or Morawetz type inequalities for super-critical *focusing*-type equations such as (1.2) and (1.18). Here we rely on a new rigidity argument, see [25, 45], that is based on the exterior energy estimates for the underlying free equation, see Proposition 6.4, and a good understanding of the underlying elliptic equations, see Lemma 6.2 and Lemma 6.3. We refer the reader to Section 6 for the details of the argument, which was inspired by the so-called *channels of energy* method developed by Duyckaerts, Kenig, and Merle in the seminal papers, [19, 20]. We emphasize that this rigidity argument does not require the use of any monotone quantities at the level of the nonlinear dynamical equation and is thus very flexible when it comes to the structure of the nonlinearity. Our implementation of this method relies crucially on the additional decay for compact trajectories proved in Section 4; see in particular Proposition 4.5 and Proposition 4.7.

In general, solutions to (1.2), (1.18) or (1.22) are only as regular and only decay as much as their initial data as evidenced by the presence of the free propagator $S(t)$ in the Duhamel representation for the solution

$$\vec{u}^*(t_0) = S(t_0 - t)\vec{u}^*(t) + \int_t^{t_0} S(t_0 - s)(0, \pm F(u^*)) ds. \quad (1.32)$$

The critical element is different however since the pre-compactness of its trajectory is at odds with the dispersive properties of the free part, $S(t_0 - t)\vec{u}^*(t)$. It follows that the first term on the right-hand-side above must vanish weakly as $t \rightarrow \sup I_{\max}, \inf I_{\max}$. The second term on the right-hand-side of (1.32) with $t = T_+$ or $t = T_-$ therefore encodes both the regularity and the spacial decay of the critical element, and in fact gains can be expected due to the presence of the nonlinear term $F(u)$. The additional regularity and decay are extracted by way of the “double Duhamel trick,” which refers to the consideration of the pairing of

$$\left\langle \int_{T_1}^{t_0} S(t_0 - s)(0, F(u)) ds, \int_{t_0}^{T_2} S(t_0 - \tau)(0, F(u)) d\tau \right\rangle$$

where $T_1 < t_0$ and $T_2 > t_0$. This technique was introduced by Colliander, Keel, Staffilani, Takaoka, Tao [7], Tao [57] and later utilized in the Kenig-Merle framework for non-linear Schrödinger equations by Killip, Visan [34, 35, 38], and for semi-linear wave equations [36, 4, 5]. This method is related to the in/out decomposition used by Killip, Tao, Visan [32, Section 6]. For more details on how to exploit the fact that differing time directions are chosen above we refer the reader to Section 4.

2. PRELIMINARIES

2.1. Some facts from harmonic analysis. We will denote by P_N the usual Littlewood-Paley projections onto frequencies of size $|\xi| \simeq N$ and by $P_{\leq N}$ the projection onto frequencies $|\xi| \lesssim N$. The frequency size N will often be a dyadic number $N = 2^k$ for some $k \in \mathbb{Z}$ and in this case we will also employ the following

abuse of notation: When we write P_k with a *lowercase* subscript k , this will mean projection onto frequencies $|\xi| \simeq 2^k$.

We recall the Bernstein inequalities.

Lemma 2.1 (Bernstein's inequalities). [56, Appendix A] *Let $1 \leq p \leq q \leq \infty$ and $s \geq 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Then*

$$\begin{aligned} \|P_{\geq N} f\|_{L^p} &\lesssim N^{-s} \|\nabla|^s P_{\geq N} f\|_{L^p}, \\ \|P_{\leq N} |\nabla|^s f\|_{L^p} &\lesssim N^s \|P_{\leq N} f\|_{L^p}, \quad \|P_N |\nabla|^{\pm s} f\|_{L^p} \simeq N^{\pm s} \|P_N f\|_{L^p} \\ \|P_{\leq N} f\|_{L^q} &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq N} f\|_{L^p}, \quad \|P_N f\|_{L^q} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p}. \end{aligned} \quad (2.1)$$

In what follows we will also require the notion of a frequency envelope.

Definition 1. [55, Definition 1] We define a *frequency envelope* to be a sequence $\beta = \{\beta_k\}$ of positive real numbers with $\beta \in \ell^2$ and

$$\|\beta\|_{\ell^2} \lesssim B.$$

If β is a frequency envelope and $(f, g) \in \dot{H}^s \times \dot{H}^{s-1}$ then we say that (f, g) *lies underneath* β if

$$\|(P_k f, P_k g)\|_{\dot{H}^s \times \dot{H}^{s-1}} \leq \beta_k \quad \forall k \in \mathbb{Z},$$

and we note that if (f, g) lie underneath β then we have

$$\|(f, g)\|_{\dot{H}^s \times \dot{H}^{s-1}} \lesssim B.$$

Next we recall a refinement of the Sobolev embedding for radially symmetric functions, which follows from the Hardy-Littlewood-Sobolev inequality.

Lemma 2.2 (Radial Sobolev Embedding). [58, Corollary A.3] *Let $0 < \gamma < 5$ and suppose $f \in \dot{W}^{\gamma, p}(\mathbb{R}^5)$ is a radial function. Suppose that*

$$\beta > -\frac{5}{q}, \quad 1 \leq \frac{1}{p} + \frac{1}{q} \leq 1 + \gamma, \quad \frac{5}{p'} + \frac{5}{q'} = 5 - \beta - \gamma$$

and at most one of the equalities $q = 1$, $q = \infty$, $p = 1$, $p = \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \gamma$, holds. Then

$$\|r^\beta f\|_{L^{q'}(\mathbb{R}^5)} \leq C \|f\|_{\dot{W}^{\gamma, p}(\mathbb{R}^5)}. \quad (2.2)$$

2.2. Strichartz estimates. For the small data theory we require Strichartz estimates for the linear wave equation in \mathbb{R}^{1+5} ,

$$\begin{aligned} v_{tt} - \Delta v &= G, \\ \vec{v}(0) &= (v_0, v_1). \end{aligned} \quad (2.3)$$

A free wave will mean a solution to (2.3) with $G = 0$ and will be denoted by $\vec{u}(t) = S(t)\vec{u}(0)$. We say that a triple (p, q, γ) is admissible if

$$p, q \geq 2, \quad \frac{1}{p} + \frac{2}{q} \leq 1, \quad \frac{1}{p} + \frac{5}{q} = \frac{5}{2} - \gamma. \quad (2.4)$$

The Strichartz estimates below are standard and we refer the reader to [24, 46] or the book [53] as well as the references therein for proofs.

Proposition 2.3. [24, 46, 53] *Let $\vec{v}(t)$ be a solution to (2.3) with initial data $\vec{v}(0) \in \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^5)$ for $s > 0$. Let (p, q, γ) , and (a, b, ρ) be admissible triples. Then, for any time interval $I \ni 0$ we have the estimates*

$$\|v\|_{L_t^p(I; W_x^{s-\gamma, q})} \lesssim \|(v_0, v_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|G\|_{L_t^{a'}(I; W_x^{s-1+\rho, b'})}. \quad (2.5)$$

2.3. Small data theory: global existence, scattering and the perturbation lemma. The usual argument based on Proposition 2.3 with $s = \frac{3}{2}$, $(p, q, \gamma) = (2, 10, 3/2)$, and $(a', b', \rho) = (1, 2, 0)$ yields the following standard small data result.

Proposition 2.4 (Small data theory). *Let $\vec{u}(0) = (u_0, u_1) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ be initial data for either (1.2), (1.18) or (1.22). Then there is a unique, solution $\vec{u}(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ on a maximal interval of existence $I_{\max}(\vec{u}) = (T_-(\vec{u}), T_+(\vec{u}))$. Moreover, for any compact interval $J \subset I_{\max}$ we have*

$$\|u\|_{L_t^2(J; L_x^{10})(\mathbb{R}^5)} < \infty.$$

A globally defined solution $\vec{u}(t)$ for $t \in [0, \infty)$ scatters as $t \rightarrow \infty$ to a free wave, i.e., a solution $\vec{u}_L(t)$ of $\square u_L = 0$ if and only if $\|u\|_{L_t^2([0, \infty), L_x^{10})} < \infty$. In particular, there exists a constant $\delta_0 > 0$ so that

$$\|\vec{u}(0)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}} < \delta_0 \implies \|u\|_{L_t^2(\mathbb{R}; L_x^{10})} \lesssim \|\vec{u}(0)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}} \lesssim \delta_0 \quad (2.6)$$

and therefore $\vec{u}(t)$ scatters to free waves as $t \rightarrow \pm\infty$. Lastly, we recall the standard finite time blow-up criterion:

$$T_+(\vec{u}) < \infty \implies \|u\|_{L_t^2([0, T_+(\vec{u})); L_x^{10})} = +\infty \quad (2.7)$$

A nearly identical statement holds when $-\infty < T_-(\vec{u})$.

Another standard result is the Perturbation Lemma for approximate solutions to (1.2), (1.18), or (1.22) which is needed in the concentration compactness procedure in Section 3.

Lemma 2.5 (Perturbation Lemma). *There are continuous functions $\varepsilon_0, C_0 : (0, \infty) \rightarrow (0, \infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded), $\vec{u}, \vec{v} \in C(I; \mathcal{H})$ satisfying for some $A > 0$*

$$\begin{aligned} & \|\vec{v}\|_{L^\infty(I; \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}})(\mathbb{R}^5)} + \|v\|_{L_t^2(I; L_x^{10})(\mathbb{R}^5)} \leq A \\ & \|\text{eq}(u)\|_{L_t^1(I; \dot{H}_x^{\frac{1}{2}})} + \|\text{eq}(v)\|_{L_t^1(I; \dot{H}_x^{\frac{1}{2}})} + \|w_0\|_{L_t^2(I; L_x^{10})} \leq \varepsilon \leq \varepsilon_0(A), \end{aligned}$$

where $\text{eq}(u)$ is either $\text{eq}(u) := \square u - u^3$, $\text{eq}(u) := \square u - Z_{\mathbb{S}^3}(ru)u^3$ or $\text{eq}(u) := \square u - Z_{\mathbb{H}^3}(ru)u^3$ in the sense of distributions, and $\vec{w}_0(t) := S(t - t_0)(\vec{u} - \vec{v})(t_0)$ with $t_0 \in I$ arbitrary but fixed. Then

$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L^\infty(I; \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}})} + \|u - v\|_{L_t^2(I; L_x^{10})} \leq C_0(A)\varepsilon.$$

In particular, $\|u\|_{S(I)} < \infty$.

3. CONCENTRATION COMPACTNESS

3.1. Existence and compactness of a critical element. We begin the proofs of Theorem 1.1 and Theorem 1.2. In both cases we follow the concentration-compactness/rigidity method introduced by Kenig and Merle in [26, 27]. The concentration compactness aspect of the argument is based on the fundamental linear and nonlinear profile decompositions of Bahouri and Gerard, [1], and is by

now standard. We essentially use the scheme from [28], which is a refinement of the techniques used in [26, 27] to extract a critical element. Indeed, the conclusion of this section is that in the event that Theorem 1.1 or Theorem 1.2 fails, there exists a minimal, nontrivial, non-scattering solution to (1.2), that is referred to as the critical element.

First some notation, and we follow [28] for convenience. Given initial data $(u_0, u_1) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ we denote by $\vec{u}(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ the unique solution to either (1.2), (1.18), or (1.22) with initial data $\vec{u}(0) = (u_0, u_1)$ defined on the maximal interval of existence $I_{\max}(\vec{u}) := (T_-(\vec{u}), T_+(\vec{u}))$. For $A > 0$ define

$$\mathcal{B}(A) := \left\{ (u_0, u_1) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}} : \|\vec{u}(t)\|_{L_t^\infty([0, T_+(\vec{u})); \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}})} \leq A \right\}.$$

Definition 2. We say that $\mathcal{SC}(A)$ holds if for all $\vec{u} = (u_0, u_1) \in \mathcal{B}(A)$ we have $T_+(\vec{u}) = +\infty$ and $\|u\|_{S([0, \infty))} < \infty$. We also say that $\mathcal{SC}(A; \vec{u})$ holds if $\vec{u} \in \mathcal{B}(A)$, $T_+(\vec{u}) = +\infty$ and $\|u\|_{S([0, \infty))} < \infty$.

Remark 3. Recall from Proposition 2.4 that $\|u\|_{S([0, \infty))} < \infty$ if and only if \vec{u} scatters to a free waves as $t \rightarrow +\infty$. It follows that both Theorem 1.1 and Theorem 1.2 are equivalent to the statement that $\mathcal{SC}(A)$ holds for all $A > 0$.

Now suppose that Theorem 1.1 (resp. Theorem 1.2) *is false*. By Proposition 2.4, there is an $A_0 > 0$ such that $\mathcal{SC}(A_0)$ holds. As we are assuming that Theorem 1.1 (resp. Theorem 1.2) fails, there exists a threshold value A_C so that for $A < A_C$, $\mathcal{SC}(A)$ holds, and for $A > A_C$, $\mathcal{SC}(A)$ fails. It is clear that $0 < A_0 < A_C$. The standard conclusion of the supposed failure of the Theorem 1.1 (resp. Theorem 1.2) is that there then must exist a non-scattering solution $\vec{u}(t)$ to (1.2) so that $\mathcal{SC}(A_C, \vec{u})$ fails, which enjoys certain minimality and compactness properties.

We state a refined version of this result below, and we refer the reader to [28, 29, 30] for the details of the argument. As usual, the main ingredients are the linear and nonlinear Bahouri-Gerard type profile decompositions from [1] used together with the nonlinear perturbation theory, Lemma 2.5.

Proposition 3.1. *Suppose that Theorem 1.1 (resp. Theorem 1.2) is false. Then, there exists a solution $\vec{u}^*(t)$, referred to as a critical element such that $\mathcal{SC}(A_C; \vec{u}^*)$ fails. Moreover, we can assume that $\vec{u}^*(t)$ does not scatter in either time direction, which means that*

$$\|u^*\|_{S((T_-(\vec{u}), 0])} = \|u^*\|_{S([0, T_+(\vec{u})))} = \infty. \quad (3.1)$$

In addition, there exists a continuous function $N : I_{\max}(\vec{u}) \rightarrow (0, \infty)$ so that the set

$$K := \left\{ \left(\frac{1}{N(t)} u^* \left(t, \frac{\cdot}{N(t)} \right), \frac{1}{N^2(t)} u_t^* \left(t, \frac{\cdot}{N(t)} \right) \right) \mid t \in I_{\max}(\vec{u}) \right\} \quad (3.2)$$

is pre-compact in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and we have

$$\inf_{t \in [0, T_+(\vec{u}))} N(t) > 0. \quad (3.3)$$

In what follows it will be convenient to give a name to the compactness property (3.2) satisfied by the critical element.

Definition 3 (The Compactness Property). Let $I \ni 0$ be a time interval and suppose $\vec{u}(t)$ be a solution to either (1.2), (1.18), or (1.22) on an interval I . We will say $\vec{u}(t)$

has the *compactness property* on I if there exists a continuous function $N : I \rightarrow (0, \infty)$ so that the set

$$K := \left\{ \left(\frac{1}{N(t)} u \left(t, \frac{\cdot}{N(t)} \right), \frac{1}{N^2(t)} u_t \left(t, \frac{\cdot}{N(t)} \right) \right) \mid t \in I \right\}$$

is pre-compact in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$.

Remark 4 (Uniformly Small Tails). A straightforward consequence of a solution having the *compactness property* on an interval I is that, after modulation, we can control the $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ tails uniformly in $t \in I$. In fact, by the Arzela-Ascoli theorem, for any $\eta_0 > 0$ there exists $0 < c(\eta_0) < C(\eta_0) < \infty$ such that

$$\begin{aligned} & \int_{|x| \geq \frac{C(\eta)}{N(t)}} \left| |\nabla|^{3/2} u(t, x) \right|^2 dx + \int_{|\xi| \geq C(\eta)N(t)} |\xi|^3 |\hat{u}(t, \xi)|^2 d\xi \leq \eta_0, \\ & \int_{|x| \leq \frac{c(\eta)}{N(t)}} \left| |\nabla|^{3/2} u(t, x) \right|^2 dx + \int_{|\xi| \leq c(\eta)N(t)} |\xi|^3 |\hat{u}(t, \xi)|^2 d\xi \leq \eta_0, \\ & \int_{|x| \geq \frac{C(\eta)}{N(t)}} \left| |\nabla|^{1/2} u_t(t, x) \right|^2 dx + \int_{|\xi| \geq C(\eta)N(t)} |\xi| |\hat{u}_t(t, \xi)|^2 d\xi \leq \eta_0, \\ & \int_{|x| \leq \frac{c(\eta)}{N(t)}} \left| |\nabla|^{1/2} u_t(t, x) \right|^2 dx + \int_{|\xi| \leq c(\eta)N(t)} |\xi| |\hat{u}_t(t, \xi)|^2 d\xi \leq \eta_0, \end{aligned} \quad (3.4)$$

for all $t \in I$.

Another standard fact about solutions to (1.2) with the compactness property on an open interval $I = (T_-, T_+)$ is that any Strichartz norm of the linear part of the evolution vanishes as $t \rightarrow T_-$ and as $t \rightarrow T_+$. A concentration compactness argument then implies that the linear part of the evolution must vanish weakly in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$, see [59, Section 6], [52, Proposition 3.6]. This implies the following lemma, which is essential to the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 3.2 (Weak Limits). [59, Section 6], [52, Proposition 3.6] *Let $\vec{u}(t)$ be a solution to either (1.2), (1.18), or (1.22) with the compactness property on an interval $I = (T_-, T_+)$. Then for any $t_0 \in I$ we have*

$$\begin{aligned} & - \int_{t_0}^T S(t_0 - s)(0, F(u)) ds \rightharpoonup \vec{u}(t_0) \quad \text{as } T \nearrow T_+ \quad \text{weakly in } \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}} \\ & + \int_T^{t_0} S(t_0 - s)(0, F(u)) ds \rightharpoonup \vec{u}(t_0) \quad \text{as } T \searrow T_- \quad \text{weakly in } \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}} \end{aligned} \quad (3.5)$$

where here $F(u)$ denotes the nonlinearity in either (1.2), (1.18), or (1.22).

3.2. Reduction to Rigidity Theorems. The proofs of Theorem 1.1 and Theorem 1.2 are now reduced to showing that a nonzero critical element, $\vec{u}^*(t)$ as in Proposition 3.1 cannot exist. We prove the following rigidity statement.

Theorem 3.3 (Rigidity Theorem). *Let $\vec{u}(t)$ be a solution to either (1.2), (1.18), or (1.22). Suppose that there exists a continuous function $N : [0, T_+(\vec{u})) \rightarrow (0, \infty)$ so that the trajectory*

$$K := \left\{ \left(\frac{1}{N(t)} u^* \left(t, \frac{\cdot}{N(t)} \right), \frac{1}{N^2(t)} u_t^* \left(t, \frac{\cdot}{N(t)} \right) \right) \mid t \in I_{\max} \right\} \quad (3.6)$$

is pre-compact in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and we have

$$\inf_{t \in [0, T_+(\vec{u}))} N(t) > 0. \quad (3.7)$$

Then $\vec{u} \equiv (0, 0)$.

Following the work of Kenig, Merle [29, Sections 5 and 6] and Duyckaerts, Kenig, and Merle [20, Section 2], the proof of Theorem 3.3 reduces to proving the following two propositions, the first where the scaling parameter $N(t)$ is bounded away from 0 on the entire interval I_{\max} , as opposed to just the half-open interval $[0, T_+)$, and the second which assumes that the solution experiences self-similar finite time blow up in forward time.

Proposition 3.4. *Let $\vec{u}(t)$ be a solution to either (1.2), (1.18), or (1.22) defined on an interval $I_{\max}(\vec{u}) = (T_-, T_+)$. Suppose that there exists a continuous function $N : I_{\max}(\vec{u}) \rightarrow (0, \infty)$ so that the trajectory*

$$K := \left\{ \left(\frac{1}{N(t)} u^* \left(t, \frac{\cdot}{N(t)} \right), \frac{1}{N^2(t)} u_t^* \left(t, \frac{\cdot}{N(t)} \right) \right) \mid t \in I_{\max}(\vec{u}) \right\} \quad (3.8)$$

is pre-compact in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and we have

$$\inf_{t \in I_{\max}(\vec{u})} N(t) > 0. \quad (3.9)$$

Then $\vec{u} \equiv (0, 0)$.

Proposition 3.5. *There is **no** radial solution to either (1.2), (1.18), or (1.22) with the compactness property on $I_{\max}(\vec{u})$ as in Theorem 3.3, with $T_+(\vec{u}) < \infty$ and with the scaling parameter given by*

$$N(t) = (T_+ - t)^{-1}$$

on the half interval $[0, T_+)$. Note that in this case the trajectory

$$K_+ := \left\{ \left((T_+ - t) u^* \left(t, (T_+ - t) \cdot \right), (T_+ - t)^2 u_t^* \left(t, (T_+ - t) \cdot \right) \right) \mid t \in [0, T_+(\vec{u})) \right\}$$

is pre-compact in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.

Remark 5. We make note of the following reductions which we will use later.

- (1) Suppose that the scaling parameter $N(t)$ as in (3.7) satisfies $N(t) \leq C < \infty$ on $[0, T_+)$ (resp. $(T_-, 0]$). Then, a standard argument, see for example [29] or [45, proof of Proposition 4.4], shows that in fact we must have $T_+ = \infty$ (resp. $T_- = -\infty$). In fact, one can then modify the profile so that $N(t) = 1$ for all $t \geq 0$, (resp. $t \leq 0$).
- (2) Let $\vec{u}(t)$ have the compactness property as in Theorem 3.3, or Proposition 3.4, 3.5. If, say, $T_+ < \infty$, then without loss of generality we can assume that $\text{supp } u(t), u_t(t) \in B(0, T_+ - t)$; see for example [29, Lemma 4.15] for more details.

Remark 6. The reduction of Theorem 3.3 to Propositions 3.4, 3.5 is a standard argument in the field and does not depend on the dimension or the precise structure of the nonlinearity. In particular, the argument in [20, Section 2], which deals with the focusing radial, supercritical semilinear wave equation in $d = 3$, applies here as well. Such reductions hold as well for different equations such as the nonlinear Schrödinger equation, see for example [33], or for the gKdV equation, see for example [31, 9].

Rather than repeat the argument that reduces Theorem 3.3 to the two propositions, we focus the rest of the paper on the proofs of Proposition 3.4, and Proposition 3.5, where several aspects of proof differ significantly from the arguments in [20].

4. ADDITIONAL REGULARITY AND DECAY FOR SOLUTIONS WITH THE COMPACTNESS PROPERTY

In this section we prove additional regularity and additional spacial decay for solutions to (1.2), (1.18), and (1.22) with the compactness property on an open interval $I_{\max} \ni 0$ using the so-called Double Duhamel trick. The methods used in this section are similar to the techniques used in [10] for the cubic wave equation in \mathbb{R}^{1+3} . In the first two subsections we carry out the arguments in the case of the focusing cubic equation (1.2) as this keeps the technical difficulties at a minimum. In the last two subsections we extend the main results to solutions to (1.18) and (1.22).

4.1. Higher regularity for compact solutions to (1.2). We begin by showing that a solution to (1.2) with that compactness property in $\dot{H}^{3/2} \times \dot{H}^{1/2}$ is in fact, more regular. In particular we prove the following estimates for the $\dot{H}^{5/2} \times \dot{H}^{3/2}$ norm of $\vec{u}(t)$.

Proposition 4.1. *Suppose $\vec{u}(t)$ is a solution to (1.2) with the compactness property on $I_{\max}(\vec{u})$ as in Theorem 3.3, i.e., assume that there exists a function $N : I_{\max} \rightarrow (0, \infty)$ so that the set*

$$K := \left\{ \left(\frac{1}{N(t)} u \left(t, \frac{\cdot}{N(t)} \right), \frac{1}{N^2(t)} u_t \left(t, \frac{\cdot}{N(t)} \right) \right) \mid t \in I \right\} \quad (4.1)$$

is pre-compact in $\dot{H}^{3/2} \times \dot{H}^{1/2}$. Then for all $t \in I_{\max}$,

$$\|\vec{u}(t)\|_{\dot{H}^{5/2} \times \dot{H}^{3/2}(\mathbb{R}^5)} \lesssim N(t).$$

with a constant that is uniform in $t \in I_{\max}$.

Remark 7. We note that all implicit constants in this section in the symbol \lesssim will be allowed to depend on the $L_t^\infty(I_{\max}; \dot{H}^{3/2} \times \dot{H}^{1/2})$ norm of \vec{u} , which is bounded by a fixed constant.

Before beginning the proof of Proposition 4.1 we establish a few preliminary definitions and facts. Define

$$v = u + \frac{i}{\sqrt{-\Delta}} u_t.$$

Note that if u solves

$$u_{tt} - \Delta u = F(u), \quad (4.2)$$

then v solves

$$v_t = u_t + \frac{i}{\sqrt{-\Delta}} (\Delta u + F(u)) = -i\sqrt{-\Delta} v + \frac{i}{\sqrt{-\Delta}} F(u).$$

and we have

$$\|\vec{u}(t)\|_{\dot{H}^s \times \dot{H}^{s-1}} \simeq \|v(t)\|_{\dot{H}^s} \quad (4.3)$$

By Duhamel's formula

$$v(t) = e^{-it\sqrt{-\Delta}}v(0) + \int_0^t \frac{e^{-i(t-\tau)\sqrt{-\Delta}}}{\sqrt{-\Delta}} F(u) d\tau.$$

If $\vec{u}(t)$ has the compactness property on I_{\max} then using Lemma 3.2 we note that for any $t_0 \in I_{\max}$, as

$$\int_{t_0}^T e^{-i(t_0-\tau)\sqrt{-\Delta}} F(u(\tau)) d\tau \rightharpoonup v(t_0), \quad \text{as } T \rightarrow T_-, T_+ \quad (4.4)$$

weakly in $\dot{H}^{3/2}(\mathbb{R}^5)$.

We next prove a refined local estimate on the scattering norm of $\vec{u}(t)$.

Lemma 4.2. *Let $\vec{u}(t)$ satisfy the assumptions in Proposition 4.1. Then for any $\eta > 0$ there exists $\delta(\eta) > 0$ such that for any $t_0 \in I_{\max}$,*

$$\|u\|_{L_t^2 L_x^{10}([t_0 - \frac{\delta}{N(t_0)}, t_0 + \frac{\delta}{N(t_0)}] \times \mathbb{R}^5)} \lesssim \eta.$$

Proof. We can assume without loss of generality that $t_0 = 0$ and define the interval $J := [-\frac{\delta}{N(t_0)}, +\frac{\delta}{N(t_0)}]$. By Duhamel's formula

$$\|u(t)\|_{L^2(J; L^{10})} \leq \|S(t)\vec{u}(0)\|_{L^2(J; L^{10})} + \left\| \int_0^t S(t-s)(0, u^3) ds \right\|_{L^2(J; L^{10})} \quad (4.5)$$

We begin by estimating the first term on the right-hand-side of (4.5). Choose $C(\eta)$ as in Remark 4, so that

$$\|P_{\geq C(\eta)N(0)}\vec{u}(0)\|_{\dot{H}^{3/2} \times \dot{H}^{1/2}(\mathbb{R}^5)} \leq \eta. \quad (4.6)$$

By compactness $C(\eta)$ above can be chosen uniformly in $t \in I_{\max}$, which is the reason why it suffices to just consider $t_0 = 0$ in this argument. Next, we have

$$\begin{aligned} \|S(t)\vec{u}(0)\|_{L_t^2(J; L_x^{10})} &\lesssim \\ &\|S(t)P_{\geq C(\eta)N(0)}\vec{u}(0)\|_{L_t^2(J; L_x^{10})} + \|S(t)P_{\leq C(\eta)N(0)}\vec{u}(0)\|_{L_t^2(J; L_x^{10})} \end{aligned}$$

We use (4.6) together with Strichartz estimates to handle the first term on the right-hand-side above:

$$\|S(t)P_{\geq C(\eta)N(0)}\vec{u}(0)\|_{L_t^2(J; L_x^{10})} \lesssim \|P_{\geq C(\eta)N(0)}\vec{u}(0)\|_{\dot{H}^{3/2} \times \dot{H}^{1/2}} \lesssim \eta.$$

To control the second term we use Bernstein's inequalities, (2.1) and Sobolev embedding,

$$\|P_{\leq C(\eta)N(0)}S(t)\vec{u}(0)\|_{L_x^{10}(\mathbb{R}^5)} \lesssim C(\eta)^{1/2} N(0)^{1/2} \quad (4.7)$$

Taking the $L_t^2(J)$ norm above yields

$$\|S(t)P_{\leq C(\eta)N(0)}\vec{u}(0)\|_{L_t^2(J; L_x^{10})} \lesssim C(\eta)^{\frac{1}{2}} \delta^{\frac{1}{2}}$$

Next, we use Strichartz estimates on the second term on the right-hand-side of (4.5).

$$\left\| \int_0^t S(t-s)(0, \pm u^3) ds \right\|_{L_t^2(J; L_x^{10})} \lesssim \|(D^{\frac{1}{2}}u)u^2\|_{L_t^1(J; L^2)} \lesssim \|u\|_{L_t^2(J; L_x^{10})}^2$$

Combining all of the above we obtain,

$$\|u\|_{L_t^2(J; L_x^{10})} \lesssim \eta + C(\eta)^{\frac{1}{2}} \delta^{\frac{1}{2}} + \|u\|_{L_t^2(J; L_x^{10})}^2 \quad (4.8)$$

The proof is concluded using the usual continuity argument after taking δ small enough. \square

We can now begin the proof of Proposition 4.1.

Proof of Proposition 4.1. Using compactness, we can assume, without loss of generality that $t_0 = 0$. We prove Proposition 4.1 by finding a frequency envelope $\alpha_k(0)$ so that

$$\begin{aligned} \|P_k \vec{u}(0)\|_{\dot{H}^{5/2} \times \dot{H}^{3/2}} &\lesssim 2^k \alpha_k(0) \\ \|\{2^k \alpha_k(0)\}_{k \in \mathbb{Z}}\|_{\ell^2} &\lesssim N(0) \end{aligned} \quad (4.9)$$

We note that finding $\alpha_k(0)$ as above proves Proposition 4.1 in light of Definition 1.

Claim 4.3. *Let $\eta > 0$ be a small number and let $J := [-\delta/N(0), +\delta/N(0)]$ where $\delta = \delta(\eta)$ is as in Lemma 4.2. Define*

$$\begin{aligned} a_k &:= 2^{\frac{3k}{2}} \|P_k u\|_{L^\infty(J; L^2)} + 2^{\frac{k}{2}} \|P_k u_t\|_{L^\infty(J; L^2)} \\ a_k(0) &:= 2^{\frac{3k}{2}} \|P_k u(0)\|_{L^2} + 2^{\frac{k}{2}} \|P_k u_t(0)\|_{L^2} \end{aligned}$$

Define the frequency envelopes

$$\alpha_k := \sum_j 2^{-\frac{5}{4}|j-k|} a_j, \quad \alpha_k(0) := \sum_j 2^{-\frac{5}{4}|j-k|} a_j(0)$$

Then,

$$a_k \lesssim a_k(0) + \eta^2 \sum_{j \geq k-3} 2^{3(k-j)/2} a_j \quad (4.10)$$

and $\eta > 0$ can be chosen small enough so that

$$\alpha_k \lesssim \alpha_k(0). \quad (4.11)$$

Proof. First we observe that after localizing to frequency k , Strichartz estimates along with Lemma 4.2 give

$$\begin{aligned} a_k &\lesssim 2^{3k/2} \|P_k u(0)\|_{L_x^2} + 2^{k/2} \|P_k u_t(0)\|_{L_x^2} + 2^{\frac{k}{2}} \|P_k(u^3)\|_{L_t^1 L_x^2(J)} \\ &\lesssim a_k(0) + \eta^2 \sum_{j \geq k-3} 2^{3(k-j)/2} a_j. \end{aligned} \quad (4.12)$$

Indeed, to prove the last line above we observe that it suffices to show that

$$2^{\frac{k}{2}} \|P_k(u^3)\|_{L_x^2} \lesssim 2^{\frac{3k}{2}} \|u\|_{L^{10}}^2 \sum_{j \geq k-3} \|P_j u\|_{L^2} \quad (4.13)$$

First, noting that since $P_k((P_{\leq k-4}u)^3) = 0$, we have

$$\begin{aligned} \|P_k u^3\|_{L_x^2} &\lesssim \|P_k[(P_{\leq k-4}u)^2 P_{\geq k-3}u]\|_{L^2} + \|P_k[(P_{\leq k-4}u)(P_{\geq k-3}u)^2]\|_{L^2} \\ &\quad + \|P_k[P_{\geq k-3}u]^3\|_{L^2} \end{aligned}$$

We estimate the terms on the right-hand side above as follows: First by Young's inequality and Bernstein's inequality,

$$\begin{aligned} \|P_k[(P_{\leq k-4}u)^2 P_{\geq k-3}u]\|_{L^2} &\lesssim \|(P_{\leq k-4}u)^2 P_{\geq k-3}u\|_{L^2} \\ &\lesssim \|P_{\leq k-4}u\|_{L^\infty}^2 \sum_{j \geq k-3} \|P_j u\|_{L^2} \\ &\lesssim 2^k \|u\|_{L^{10}}^2 \sum_{j \geq k-3} \|P_j u\|_{L^2} \end{aligned}$$

Next, again using Young's inequality and the fact that P_k is given by convolution with $\check{\phi}_k(\cdot) := 2^{5k}\check{\phi}(2^k\cdot)$ where $\check{\phi} \in \mathcal{S}$, we have

$$\begin{aligned} \|P_k[(P_{\leq k-4}u)(P_{\geq k-3}u)^2]\|_{L^2} &\lesssim \|P_{\leq k-4}u\|_{L^\infty} \|\check{\phi}_k\|_{L^{\frac{10}{9}}} \|(P_{\geq k-3}u)^2\|_{L^{\frac{5}{3}}} \\ &\lesssim 2^k \|u\|_{L^{10}}^2 \sum_{j \geq k-3} \|P_j u\|_{L^2} \end{aligned}$$

Finally, arguing similarly for the last term we have

$$\begin{aligned} \|P_k[P_{\geq k-3}u]^3\|_{L^2} &\lesssim \|\check{\phi}_k\|_{L^{\frac{5}{4}}} \|(P_{\geq k-3}u)^2 P_{\geq k-3}u\|_{L^{\frac{10}{7}}} \\ &\lesssim 2^k \|u\|_{L^{10}}^2 \sum_{j \geq k-3} \|P_j u\|_{L^2} \end{aligned}$$

This proves (4.13) as desired, which implies (4.10). To establish (4.11) we use (4.10) to obtain

$$\sum_j 2^{-\frac{5}{4}|j-k|} a_j \lesssim \sum_j a_j(0) 2^{-\frac{5}{4}|j-k|} + \eta^2 \sum_j 2^{-\frac{5}{4}|j-k|} \sum_{j_1 \geq j-3} 2^{3(j-j_1)/2} a_{j_1}. \quad (4.14)$$

Reversing the order of summation in the second term above we have

$$\begin{aligned} \sum_{j_1 \leq k} \sum_{j \leq j_1+3} 2^{\frac{3}{2}(j-j_1)} 2^{\frac{5}{4}(j-k)} a_{j_1} &\lesssim \sum_{j_1 \leq k} 2^{\frac{5}{4}(j_1-k)} a_{j_1} \lesssim \alpha_k, \\ \sum_{j_1 > k} \sum_{j \leq j_1+3} 2^{\frac{3}{2}(j-j_1)} 2^{-\frac{5}{4}|j-k|} a_{j_1} &\lesssim \sum_{j_1 > k} (2^{-\frac{3}{2}(j_1-k)} + 2^{-\frac{5}{4}(j_1-k)}) a_{j_1} \lesssim \alpha_k. \end{aligned} \quad (4.15)$$

Therefore, (4.14) implies that

$$\alpha_k \lesssim \alpha_k(0) + \eta^2 \alpha_k,$$

which in turn yields (4.11) once $\eta > 0$ is chosen small enough. \square

Now we are ready to use the double Duhamel trick to prove Proposition 4.1. Let $\langle \cdot, \cdot \rangle_{\dot{H}^{3/2}}$ denote the inner product with

$$\langle v, v \rangle_{\dot{H}^{3/2}} = \|v\|_{\dot{H}^{3/2}}^2.$$

Again, recall that we can assume without loss of generality that $t_0 = 0$. For any k , and for any $T_1 < T_+$ we have

$$\begin{aligned} \langle P_k v(0), P_k v(0) \rangle_{\dot{H}^{3/2}} &= \left\langle P_k \left(e^{-iT_1 \sqrt{-\Delta}} v(T_1) + \int_0^{T_1} \frac{e^{-i\tau \sqrt{-\Delta}}}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right), P_k v(0) \right\rangle_{\dot{H}^{3/2}} \\ &= \lim_{T_1 \rightarrow T_+} \left\langle P_k \left(\int_0^{T_1} \frac{e^{-i\tau \sqrt{-\Delta}}}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right), P_k v(0) \right\rangle_{\dot{H}^{3/2}}, \end{aligned}$$

where $F(u) = u^3$. Then for any $T_- < T_2 < 0$, this further reduces as

$$\begin{aligned} &= \left\langle P_k v(0), P_k e^{-iT_2 \sqrt{-\Delta}} v(T_2) \right\rangle_{\dot{H}^{3/2}} \\ &+ \lim_{T_1 \rightarrow T_+} \left\langle P_k \left(\int_0^{T_1} \frac{e^{-i\tau \sqrt{-\Delta}}}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right), P_k \left(\int_{T_2}^0 \frac{e^{-it \sqrt{-\Delta}}}{\sqrt{-\Delta}} F(u(t)) dt \right) \right\rangle_{\dot{H}^{3/2}} \end{aligned}$$

Taking the limit $T_2 \rightarrow T_-$, for any integer k , yields

$$\begin{aligned} \langle P_k v(0), P_k v(0) \rangle_{\dot{H}^{3/2}} &= \\ &= \left\langle P_k \left(\int_0^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right), P_k \left(\int_{T_-}^0 \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} F(u(t)) dt \right) \right\rangle_{\dot{H}^{3/2}}. \end{aligned}$$

Now let χ be a smooth, radial, non-increasing function, $\chi(x) = 1$ when $|x| \leq 1$, $\chi(x) = 0$ when $|x| \geq 2$. Let $c > 0$ be a small fixed constant, say $c = \frac{1}{4}$. We rewrite the inner product above as

$$\begin{aligned} \langle A + B, A' + B' \rangle_{\dot{H}^{3/2}} &= \langle A, A' + B' \rangle_{\dot{H}^{3/2}} + \langle A + B, A' \rangle_{\dot{H}^{3/2}} \\ &\quad - \langle A, A' \rangle_{\dot{H}^{3/2}} + \langle B, B' \rangle_{\dot{H}^{3/2}}, \end{aligned} \quad (4.16)$$

where

$$\begin{aligned} A &:= P_k \left(\int_0^{\frac{\delta}{N(0)}} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3 dt \right) + P_k \left(\int_{\frac{\delta}{N(0)}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi)\left(\frac{x}{ct}\right) u^3 dt \right), \\ B &:= P_k \left(\int_{\frac{\delta}{N(0)}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} \chi\left(\frac{x}{ct}\right) u^3 dt \right). \end{aligned}$$

and A', B' are the corresponding integrals in the negative time direction.

First we estimate the term $\langle A, A' \rangle_{\dot{H}^{3/2}} \leq \|A\|_{\dot{H}^{3/2}} \|A'\|_{\dot{H}^{3/2}}$. To control the terms in the right-hand side in the inequality in the preceding line we observe first that by the argument in Claim 4.3 we have

$$\left\| P_k \left(\int_0^{\frac{\delta}{N(0)}} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3 dt \right) \right\|_{\dot{H}^{3/2}} \lesssim \eta^2 \sum_{j \geq 3} 2^{\frac{3}{2}(k-j)} a_j. \quad (4.17)$$

Next, we prove the following claim:

Claim 4.4. *There exists a sequence $b_k \in \ell^2$, so that*

$$\left\| P_k \left(\int_{\frac{\delta}{N(0)}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi)\left(\frac{x}{ct}\right) u^3 dt \right) \right\|_{\dot{H}^{3/2}} \lesssim 2^{-k} b_k. \quad (4.18)$$

and

$$\|b_k\|_{\ell^2} \lesssim \frac{N(0)}{c^2 \delta} \|u\|_{L_t^\infty \dot{H}^{3/2}}^3, \quad (4.19)$$

Proof of Claim 4.4. Note that it suffices to show that

$$\left\| \int_{\frac{\delta}{N(0)}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi)\left(\frac{x}{ct}\right) u^3 dt \right\|_{\dot{H}^{5/2}} \lesssim \frac{N(0)}{c^2 \delta} \|u\|_{L_t^\infty \dot{H}^{3/2}}^3. \quad (4.20)$$

To see this we begin by observing that

$$\begin{aligned} \left\| \int_{\frac{\delta}{N(0)}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi)\left(\frac{x}{ct}\right) u^3 dt \right\|_{\dot{H}^{5/2}} &\lesssim \int_{\frac{\delta}{N(0)}}^\infty \|(1 - \chi)\left(\frac{x}{ct}\right) u^3\|_{\dot{H}^{\frac{3}{2}}} dt \\ &\lesssim \int_{\frac{\delta}{N(0)}}^\infty \left[\frac{1}{ct} \|\chi'\left(\frac{x}{ct}\right) u^3\|_{\dot{H}^{\frac{1}{2}}} + \|(1 - \chi)\left(\frac{x}{ct}\right) \nabla u u^2\|_{\dot{H}^{\frac{1}{2}}} \right] dt \end{aligned} \quad (4.21)$$

Next, using Sobolev embedding we have

$$\begin{aligned}
\frac{1}{ct} \|\chi'(\frac{x}{ct})u^3\|_{\dot{H}^{\frac{1}{2}}} &\lesssim \frac{1}{ct} \|\chi'(\frac{x}{ct})u^3\|_{\dot{W}^{1, \frac{5}{3}}} \\
&\lesssim \frac{1}{(ct)^2} \|u\|_{L^5}^3 + \frac{1}{ct} \|\chi'(\frac{x}{ct})\nabla u u^2\|_{L^{\frac{5}{3}}} \\
&\lesssim \frac{1}{(ct)^2} \|u\|_{\dot{H}^{\frac{3}{2}}}^3 + \frac{1}{ct} \|\nabla u\|_{L^{\frac{5}{2}}} \|u\|_{L^5} \|u\|_{L^\infty(|x|\geq ct)} \\
&\lesssim \frac{1}{(ct)^2} \|u\|_{\dot{H}^{\frac{3}{2}}}^3
\end{aligned}$$

where in the last line above we have used Lemma 2.2, i.e.,

$$\|u\|_{L_x^\infty(|x|\geq R)} \lesssim R^{-1} \|u\|_{\dot{H}^{3/2}(\mathbb{R}^5)}. \quad (4.22)$$

By the fractional product rule

$$\begin{aligned}
\|(1-\chi)(\frac{x}{tc})(\nabla u)u^2\|_{\dot{H}^{1/2}} &\lesssim \|\nabla u\|_{L_x^{5/2}} \|(1-\chi)(\frac{x}{tc})u^2\|_{\dot{W}^{1/2, 10}} \\
&\quad + \|u\|_{\dot{H}^{3/2}} \|(1-\chi)(\frac{x}{tc})u^2\|_{L_x^\infty}.
\end{aligned} \quad (4.23)$$

Again, by Sobolev embedding and Lemma 2.2 we have

$$\begin{aligned}
\|(1-\chi)(\frac{x}{ct})u^2\|_{\dot{W}^{1/2, 10}} &\lesssim \|(1-\chi)(\frac{x}{ct})u^2\|_{\dot{W}^{1, 5}} \\
&\lesssim \frac{1}{ct} \|u\|_{L^\infty(|x|\geq ct)} \|u\|_{L^5} + \|(1-\chi)(\frac{x}{ct})u\nabla u\|_{L^5} \\
&\lesssim \frac{1}{(ct)^2} \|u\|_{\dot{H}^{\frac{3}{2}}}^3 + \|(1-\chi)(\frac{x}{ct})u\|_{L^{10}} \|(1-\chi)(\frac{x}{ct})\nabla u\|_{L^{10}} \\
&\lesssim \frac{1}{(ct)^2} \|u\|_{\dot{H}^{\frac{3}{2}}}^3
\end{aligned}$$

We also can use Lemma 4.22 to deduce that

$$\|(1-\chi)(\frac{x}{tc})u^2\|_{L_x^\infty} \lesssim \frac{1}{(ct)^2} \|u\|_{\dot{H}^{\frac{3}{2}}}^2$$

Therefore we can estimate (4.23) by

$$\|(1-\chi)(\frac{x}{tc})(\nabla u)u^2\|_{\dot{H}^{1/2}} \lesssim \frac{1}{(ct)^2} \|u\|_{\dot{H}^{\frac{3}{2}}}^3 \quad (4.24)$$

Plugging the preceding estimates into (4.21) gives

$$\left\| \int_{\frac{\delta}{N(0)}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1-\chi)(\frac{x}{ct})u^3 dt \right\|_{\dot{H}^{5/2}} \lesssim \int_{\frac{\delta}{N(0)}}^{\infty} \frac{1}{(ct)^2} \|u\|_{\dot{H}^{\frac{3}{2}}}^3 dt \quad (4.25)$$

from which (4.20) is immediate. \square

Combining (4.17) and (4.18) and recalling the definition of A yields

$$\|A\|_{\dot{H}^{3/2}} \lesssim 2^{-k} b_k + \eta^2 \sum_{j \geq k-3} a_j 2^{3(k-j)/2}.$$

Estimating A' in an identical manner then yields

$$\langle A, A' \rangle_{\dot{H}^{\frac{3}{2}}} \lesssim \left(2^{-k} b_k + \eta^2 \sum_{j \geq k-3} a_j 2^{3(k-j)/2} \right)^2. \quad (4.26)$$

Next using again the fact that $e^{-it\sqrt{-\Delta}}v(t) \rightharpoonup 0$ weakly in $\dot{H}^{3/2}$ as $t \nearrow T_+$, $t \searrow T_-$, as well as (4.17) and (4.18) we have

$$\langle A + B, A' \rangle + \langle A, A' + B' \rangle \lesssim a_k(0) \left(2^{-k} b_k + \eta^2 \sum_{j \geq k-3} a_j 2^{3(k-j)/2} \right). \quad (4.27)$$

Finally it remains to estimate $\langle B, B' \rangle$ which is given by

$$\begin{aligned} \langle B, B' \rangle_{\dot{H}^{3/2}} &= \\ &= \int_{T_-}^{\frac{-\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^{T_+} \left\langle P_k \left(\frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} \chi\left(\frac{\cdot}{ct}\right) u^3(t) \right), P_k \left(\frac{e^{-i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} \chi\left(\frac{\cdot}{c\tau}\right) u^3(\tau) \right) \right\rangle_{\dot{H}^{\frac{3}{2}}} dt d\tau \\ &= \int_{-T_-}^{\frac{-\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^{T_+} \left\langle \chi\left(\frac{x}{ct}\right) u^3(t), P_k^2 \left(e^{i(t-\tau)\sqrt{-\Delta}} \sqrt{-\Delta} \chi\left(\frac{\cdot}{c\tau}\right) u^3(\tau) \right) \right\rangle_{L^2} dt d\tau. \end{aligned} \quad (4.28)$$

Here we use an argument based on the sharp Huygens principle, see for example [10, Section 4]. The kernel $K_k(\cdot)$ of the operator $P_k e^{i(t-\tau)\sqrt{-\Delta}} \sqrt{-\Delta}$ is given by

$$K_k(x) = K_k(|x|) := c 2^k \int_0^\pi \int_0^\infty e^{i|x|\rho \cos \theta} e^{i\rho(t-\tau)} \phi\left(\frac{\rho}{2^k}\right) \rho^5 d\rho \sin^3 \theta d\theta \quad (4.29)$$

where the integrand is written in polar coordinates on \mathbb{R}^5 where $\rho = |\xi|$, θ is the azimuthal angle on \mathbb{S}^4 , and $\phi(\cdot/2^k)$ is the Fourier multiplier for the k th Littlewood-Paley projection P_k . Integration by parts $L \in \mathbb{N}$ times in ρ yields the estimate

$$|K_k(|x-y|)| \lesssim_L \frac{2^{6k}}{\langle 2^k |(\tau-t) - |x-y|| \rangle^L}$$

where here recall that $\tau > 0$ and $t < 0$. Note that in (4.28) we have $|x| \leq \frac{1}{4}|t|$ and $|y| \leq \frac{1}{4}|\tau|$ which means that $|x-y| \leq \frac{1}{4}|t-\tau|$. Therefore we have

$$(\tau-t) - |x-y| \geq \frac{1}{2}|\tau-t|$$

which yields the estimate

$$|K_k(|x-y|)| \lesssim_L \frac{2^{6k}}{\langle 2^k |\tau-t| \rangle^L} \quad (4.30)$$

in the relevant region of integration. First, if $N(0) \ll 2^k$, we use (4.30) with $L = 9$ which yields,

$$\begin{aligned} &\left\langle \chi\left(\frac{x}{ct}\right) u^3(t), P_k^2 \left(e^{i(t-\tau)\sqrt{-\Delta}} \sqrt{-\Delta} \chi\left(\frac{\cdot}{c\tau}\right) u^3(\tau) \right) \right\rangle_{L^2} \\ &\lesssim \int_0^{c|t|} u^3(t, x) \int_0^{c|\tau|} |K_k(x-y)| u^3(\tau, y) dy dx \\ &\lesssim 2^{-3k} \frac{|\tau|^2 |t|^2}{|\tau-t|^9} \|u\|_{L_t^\infty L_x^5}^6 \lesssim \frac{2^{-3k}}{|\tau-t|^5} \|u\|_{L_t^\infty \dot{H}_x^{\frac{3}{2}}}^6 \end{aligned} \quad (4.31)$$

Integrating (4.31) in τ and in t as in (4.28) gives

$$\begin{aligned} \langle B, B' \rangle_{\dot{H}^{\frac{3}{2}}} &\lesssim \int_{-T_-}^{\frac{-\delta}{N(0)}} \int_{\frac{\delta}{N(0)}}^{T_+} \frac{2^{-3k}}{|\tau - t|^5} d\tau dt \\ &\lesssim 2^{-3k} N(0)^3 \lesssim 2^{-\frac{9}{4}k} N(0)^{\frac{9}{4}} \quad \text{if } N(0) \ll 2^k \end{aligned} \quad (4.32)$$

where in the last inequality above we have used that $N(0) \ll 2^k$. Next suppose that $2^k \lesssim N(0)$. In the region $|\tau - t| \leq 2^{-k}$ we used the crude estimate $|K_k(x - y)| \lesssim 2^{6k}$ and in the region $|\tau - t| \geq 2^{-k}$ we use (4.30) with $L = 7$ to obtain, via the same argument as above, that

$$\langle B, B' \rangle_{\dot{H}^{\frac{3}{2}}} \lesssim 1 \quad \text{if } N(0) \gtrsim 2^k \quad (4.33)$$

Putting (4.32) and (4.33) together gives

$$\langle B, B' \rangle_{\dot{H}^{\frac{3}{2}}} \lesssim \min(2^{-\frac{9}{4}k} N(0)^{\frac{9}{4}}, 1) \quad (4.34)$$

Combining (4.26), (4.27), and (4.34) gives

$$\begin{aligned} a_k^2(0) &\lesssim \left(2^{-k} b_k + \eta^2 \sum_{j \geq k-3} 2^{3(k-j)/2} a_j \right)^2 + a_k(0) \left(2^{-k} b_k + \eta^2 \sum_{j \geq k-3} a_j 2^{3(k-j)/2} \right) \\ &\quad \min(2^{-\frac{9}{4}k} N(0)^{\frac{9}{4}}, 1) \end{aligned}$$

which implies that

$$a_k(0) \lesssim 2^{-k} b_k + \eta^2 \sum_{j \geq k-3} 2^{3(k-j)/2} a_j + \min(2^{-\frac{9}{8}k} N(0)^{\frac{9}{8}}, 1) \quad (4.35)$$

Recalling the definitions of the envelopes $\alpha_k(0)$ and α_k we have

$$\alpha_k(0) \lesssim \eta^2 \alpha_k + \sum_j 2^{-\frac{5}{4}|j-k|} 2^{-j} b_j + \sum_j 2^{-\frac{5}{4}|j-k|} 2^{-j} \min(2^{-\frac{1}{8}j} N(0)^{\frac{9}{8}}, 2^j) \quad (4.36)$$

Next, using (4.11) to control $\alpha_k \lesssim \alpha_k(0)$, and taking $\eta > 0$ small enough we obtain

$$\alpha_k(0) \lesssim \sum_j 2^{-\frac{5}{4}|j-k|} 2^{-j} b_j + \sum_j 2^{-\frac{5}{4}|j-k|} 2^{-j} c_j \quad (4.37)$$

where $c_j := \min(2^{-\frac{1}{8}j} N(0)^{\frac{9}{8}}, 2^j)$ satisfies

$$\|\{c_j\}\|_{\ell^2} \lesssim N(0) \quad (4.38)$$

Finally, we use Schur's test along with (4.19), (4.38) to deduce that

$$\|\{2^k \alpha_k\}\|_{\ell^2} \lesssim N(0) \quad (4.39)$$

as desired. This completes the proof of Proposition 4.1 \square

4.2. Improved uniform spacial decay for compact solutions to (1.2). In this section we prove that in the case that $\inf_{t \in I} N(t) > 0$ then a solution $\vec{u}(t)$ to (1.2) with the compactness property on I has uniform spacial decay that “breaks the scaling.” In particular, we show that $\vec{u}(t)$ must be uniformly bounded in $\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}(\mathbb{R}^5)$, which in turn implies by Lemma 2.2 that $|r^{7/4}u(t, r)| \lesssim 1$. We will not use this pointwise estimate in particular, but rather the fact that boundedness in $\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}(\mathbb{R}^5)$ and compactness in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^5)$ will yield compactness in the energy space $\dot{H}^1 \times L^2$. This last point will be crucial in Section 6.

Proposition 4.5. *Let $\vec{u}(t)$ be a solution to (1.2) with the Compactness Property as in Proposition 3.4, i.e, suppose that the scaling parameter $N(t)$ satisfies*

$$\inf_{t \in I_{\max}(\vec{u})} N(t) > 0$$

then for all $t \in I_{\max}$,

$$\|\vec{u}(t)\|_{\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}(\mathbb{R}^5)} \lesssim 1. \quad (4.40)$$

with a constant that is uniform in $t \in I_{\max}$.

Proof. Without loss of generality, it will again suffice to consider only the case $t = 0$. We will prove Proposition 4.5 by finding a frequency envelope $\alpha_k = \alpha_k(0)$ so that

$$\begin{aligned} \|(P_k u(0), P_k u_t(0))\|_{\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}} &\lesssim 2^{-\frac{3k}{4}} \alpha_k \\ \|\{2^{-\frac{3k}{4}} \alpha_k\}_{k \in \mathbb{Z}}\|_{\ell^2} &\lesssim 1 \end{aligned} \quad (4.41)$$

Note that for $k \geq 0$ it suffices, by the uniform boundedness of the $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ norm of $\vec{u}(t)$ to take

$$\alpha_k := 1 \quad \text{for } k \geq 0$$

Now, for each j define

$$a_j := 2^{3j/2} \|P_j u\|_{L_t^\infty L_x^2} + 2^{j/2} \|P_j u_t\|_{L_t^\infty L_x^2} \quad (4.42)$$

and for $k < 0$ set

$$\alpha_k := \sum_j 2^{-|j-k|} a_j \quad \text{for } k < 0 \quad (4.43)$$

As in the previous subsection let $v = u + \frac{i}{\sqrt{-\Delta}} u_t$. Then v solves the equation

$$v_t = u_t + \frac{i}{\sqrt{-\Delta}} u_{tt} = i u_t + \frac{i}{\sqrt{-\Delta}} (\Delta u + u^3) = -i \sqrt{-\Delta} v + \frac{i}{\sqrt{-\Delta}} u^3.$$

and we have

$$\|P_j v\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \simeq a_j \quad (4.44)$$

Next, note that we can also assume without loss of generality that $N(t) \geq 1$ for all $t \in I_{\max}$. By Lemma 3.2 we have the weak limits

$$\lim_{t \nearrow T_+, t \searrow T_-} e^{it\sqrt{-\Delta}} v(t) \rightharpoonup 0,$$

weakly in $\dot{H}^{3/2}$. As in the previous subsection we thus obtain the reduction

$$\begin{aligned} \langle P_M v(0), P_M v(0) \rangle_{\dot{H}^{3/2}} &= \\ &= \left\langle P_M \left(\int_0^{T_+} \frac{e^{-i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3(\tau) d\tau \right), P_M \left(\int_{T_-}^0 \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3(t) dt \right) \right\rangle_{\dot{H}^{3/2}}. \end{aligned}$$

for any fixed frequency M . As we are only concerned with low frequencies in what follows $M < 1$ and later we will write $M = 2^k$ for $k < 0$.

Again as in the previous subsection we let χ be a smooth, radial, non-increasing function, $\chi(x) = 1$ when $|x| \leq 1$, $\chi(x) = 0$ when $|x| \geq 2$. Let $c > 0$ be a small fixed constant, say $c = \frac{1}{4}$. We rewrite the inner product above as

$$\begin{aligned} \langle A + B, A' + B' \rangle_{\dot{H}^{3/2}} &= \langle A, A' + B' \rangle_{\dot{H}^{3/2}} + \langle A + B, A' \rangle_{\dot{H}^{3/2}} \\ &\quad - \langle A, A' \rangle_{\dot{H}^{3/2}} + \langle B, B' \rangle_{\dot{H}^{3/2}}, \end{aligned} \quad (4.45)$$

where

$$\begin{aligned} A &:= \int_0^{CM^{-1}} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} P_M(u^3(t)) dt + \int_{CM^{-1}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1 - \chi)\left(\frac{x}{ct}\right) P_M(u^3(t)) dt, \\ B &:= \int_{CM^{-1}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} \chi\left(\frac{x}{ct}\right) P_M(u^3(t)) dt. \end{aligned}$$

and C is will be a fixed constant to be determined below. Again, A' , B' are the corresponding integrals in the negative time direction.

First we estimate the term $\langle A, A' \rangle_{\dot{H}^{3/2}} \leq \|A\|_{\dot{H}^{3/2}} \|A'\|_{\dot{H}^{3/2}}$. First, we have

$$\begin{aligned} \left\| P_M \left(\int_0^{CM^{-1}} \frac{e^{-i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3(\tau) d\tau \right) \right\|_{\dot{H}^{\frac{3}{2}}} &\lesssim M^{\frac{1}{2}} \left\| \int_0^{CM^{-1}} e^{-i\tau\sqrt{-\Delta}} P_M u^3(\tau) d\tau \right\|_{L^2} \\ &\lesssim CM^{-\frac{1}{2}} \|P_M u^3\|_{L_t^\infty L_x^2} \end{aligned} \quad (4.46)$$

We also note that by the compactness of the set K as in Proposition 3.4 and using the fact that $N(t) \geq 1$ we can find a small number $N_0 = N_0(\eta)$ so that

$$\|P_{\leq N_0} u\|_{L^5} \lesssim \|P_{\leq N_0} u\|_{\dot{H}^{\frac{3}{2}}} \lesssim \eta \quad (4.47)$$

where we have used Sobolev embedding and Remark 4 above. We now estimate the right-hand-side of (4.46). We claim that

$$CM^{-\frac{1}{2}} \|P_M u^3\|_{L_t^\infty L_x^2} \lesssim C\eta^2 M^{\frac{3}{2}} \|P_{>M/4} u\|_{L_t^\infty L^2} + CN_0^{-\frac{3}{2}} M^{\frac{3}{2}} \quad (4.48)$$

where the implicit constants above are allowed, as always, to depend on $\|v\|_{L_t^\infty \dot{H}^{\frac{3}{2}}}$ and the constant C is yet to be chosen. To prove (4.48) we write

$$\begin{aligned} \|P_M(u^3)\|_{L^2} &= \|P_M[(P_{\leq M/4} u + P_{>M/4} u)(P_{\leq N_0} u + P_{>N_0} u)^2]\|_{L^2} \\ &\lesssim \|P_M(P_{\leq M/4} u(P_{\leq N_0} u)^2)\|_{L^2} + \|P_M(P_{\leq M/4} u P_{\leq N_0} u P_{>N_0} u)\|_{L^2} \\ &\quad + \|P_M(P_{\leq M/4} u(P_{>N_0} u)^2)\|_{L^2} + \|P_M(P_{>M/4} u(P_{\leq N_0} u)^2)\|_{L^2} \\ &\quad + \|P_M(P_{>M/4} u P_{\leq N_0} u P_{>N_0} u)\|_{L^2} + \|P_M(P_{>M/4} u(P_{>N_0} u)^2)\|_{L^2} \end{aligned}$$

We next estimate each term on the right-hand-side above. We begin with the term $\|P_M(P_{\leq M/4} u(P_{\leq N_0} u)^2)\|_{L^2}$. If $N_0 \leq M/4$ then this term is identically zero. In the case that $M/4 < N_0$ we write $P_{\leq N_0} u = P_{\leq M/4} u + P_{M/4 < \cdot \leq N_0} u$ and we have

$$\begin{aligned} \|P_M(P_{\leq M/4} u(P_{\leq N_0} u)^2)\|_{L^2} &\lesssim \|P_M((P_{\leq M/4} u)^2 P_{M/4 < \cdot \leq N_0} u)\|_{L^2} \\ &\quad + \|P_M(P_{\leq M/4} u (P_{M/4 < \cdot \leq N_0} u)^2)\|_{L^2} \end{aligned}$$

recalling that P_M is given by convolution with $\check{\phi}_M = M^5 \check{\phi}(M \cdot)$, $\check{\phi} \in \mathcal{S}$, we use Young's inequality followed by Hölder to obtain

$$\begin{aligned} \|P_M((P_{\leq M/4} u)^2 P_{M/4 < \cdot \leq N_0} u)\|_{L^2} &\lesssim \|\check{\phi}_M\|_{L^{\frac{5}{3}}} \|P_{\leq M/4} u\|_{L^5}^2 \|P_{M/4 < \cdot \leq N_0} u\|_{L^2} \\ &\lesssim M^2 \eta^2 \|P_{>M/4} u\|_{L^2} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|P_M(P_{\leq M/4}u(P_{M/4 < \cdot \leq N_0}u)^2)\|_{L^2} &\lesssim \\ &\lesssim \|\check{\phi}_M\|_{L^{\frac{5}{3}}} \|P_{\leq M/4}u\|_{L^5} \|P_{M/4 < \cdot \leq N_0}u\|_{L^5} \|P_{M/4 < \cdot \leq N_0}u\|_{L^2} \\ &\lesssim M^2 \eta^2 \|P_{> M/4}u\|_{L^2} \end{aligned}$$

where we have been using throughout the fact that $M/4 < N_0$. Next we show how to control the term $\|P_M(P_{> M/4}u(P_{> N_0}u)^2)\|_{L^2}$. Using the same argument as above, together with Sobolev embedding we have

$$\begin{aligned} \|P_M(P_{> M/4}u(P_{> N_0}u)^2)\|_{L^2} &\lesssim \|\check{\phi}_M\|_{L^{\frac{5}{3}}} \|P_{> M/4}u\|_{L^5} \|P_{> N_0}u\|_{L^5} \|P_{> N_0}u\|_{L^2} \\ &\lesssim M^2 \|u\|_{\dot{H}^{\frac{3}{2}}}^2 \|P_{> N_0}u\|_{L^2} \lesssim M^2 N_0^{-\frac{3}{2}} \end{aligned}$$

The remaining terms are all handled similarly as the above two are representative and we thus obtain (4.48). Combining (4.48) with (4.46) gives

$$\left\| P_M \left(\int_0^{CM^{-1}} \frac{e^{-i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3(\tau) d\tau \right) \right\|_{\dot{H}^{\frac{3}{2}}} \lesssim C \eta^2 M^{\frac{3}{2}} \|P_{> M/4}u\|_{L_t^\infty L^2} + C N_0^{-\frac{3}{2}} M^{\frac{3}{2}} \quad (4.49)$$

We next control the second term in A using the Lemma 2.2 to obtain sufficient time decay. Indeed, we have

$$\begin{aligned} &\left\| \int_{CM^{-1}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1-\chi)\left(\frac{x}{ct}\right) P_M(u^3(t)) dt \right\|_{\dot{H}^{\frac{3}{2}}} \\ &\lesssim \int_{CM^{-1}}^\infty \left\| (1-\chi)\left(\frac{x}{ct}\right) P_M u^3 \right\|_{\dot{H}^{\frac{1}{2}}} dt \lesssim \int_{CM^{-1}}^\infty \left\| (1-\chi)\left(\frac{x}{ct}\right) P_M u^3 \right\|_{\dot{W}^{1, \frac{5}{3}}} dt \quad (4.50) \end{aligned}$$

We estimate the integrand on the right-hand-side above.

$$\left\| (1-\chi)\left(\frac{x}{ct}\right) P_M u^3 \right\|_{\dot{W}^{1, \frac{5}{3}}} \lesssim \frac{1}{ct} \left\| \chi'\left(\frac{x}{ct}\right) P_M u^3 \right\|_{L^{\frac{5}{3}}} + \left\| (1-\chi)\left(\frac{x}{ct}\right) \nabla P_M u^3 \right\|_{L^{\frac{5}{3}}} \quad (4.51)$$

Using Lemma 2.2 on the first term on the right in (4.51) gives

$$\begin{aligned} \frac{1}{ct} \left\| \chi'\left(\frac{x}{ct}\right) P_M u^3 \right\|_{L^{\frac{5}{3}}} &\lesssim \frac{1}{(ct)^2} \|r P_M u^3\|_{L^{\frac{5}{3}}} \lesssim \frac{1}{(ct)^2} \|\nabla|^{\frac{1}{2}} P_M u^3\|_{L^{\frac{10}{9}}} \\ &\lesssim \frac{1}{(ct)^2} M^{\frac{1}{2}} \|P_M u^3\|_{L^{\frac{10}{9}}} \lesssim \frac{1}{(ct)^2} M^{\frac{1}{2}} \|P_{> M/4}u\|_{L^2} \|u\|_{L^5}^2 \end{aligned} \quad (4.52)$$

where in the very last line we used Young's inequality and the decomposition $u = P_{\leq M/4}u + P_{> M/4}u$. We again use Lemma 2.2 to estimate the second term in (4.51).

$$\begin{aligned} \left\| (1-\chi)\left(\frac{x}{ct}\right) \nabla P_M u^3 \right\|_{L^{\frac{5}{3}}} &\lesssim (ct)^{-\frac{6}{5}} \|r^{\frac{6}{5}} \nabla P_M u^3\|_{L^{\frac{5}{3}}} \\ &\lesssim (ct)^{-\frac{6}{5}} \|\nabla|^{\frac{3}{10}} \nabla P_M u^3\|_{L^{\frac{10}{9}}} \lesssim (ct)^{-\frac{6}{5}} M^{\frac{13}{10}} \|\nabla P_M u^3\|_{L^{\frac{10}{9}}} \\ &\lesssim (ct)^{-\frac{6}{5}} M^{\frac{13}{10}} \|P_{> M/4}u\|_{L^2} \|u\|_{L^5}^2 \end{aligned} \quad (4.53)$$

Plugging the estimates (4.52) and (4.53) into the integrand in (4.50) and integrating in t from CM^{-1} to ∞ gives

$$\begin{aligned} \left\| \int_{CM^{-1}}^{T_+} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1-\chi)\left(\frac{x}{ct}\right) P_M(u^3(t)) dt \right\|_{\dot{H}^{\frac{3}{2}}} &\lesssim \\ &\lesssim \left(\frac{1}{C} + \frac{1}{C^{\frac{1}{5}}}\right) M^{\frac{3}{2}} \|P_{>M/4} u\|_{L_t^\infty L^2} \|u\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \end{aligned} \quad (4.54)$$

Combining (4.49) and (4.54) and choosing the constant $C := \frac{1}{\eta} > 1$ (where η remains to be fixed below) we obtain,

$$\begin{aligned} \|A\|_{\dot{H}^{\frac{3}{2}}} &\lesssim (C\eta^2 + C^{-\frac{1}{5}} + C^{-1}) M^{\frac{3}{2}} \|P_{>M/4} u\|_{L_t^\infty L^2} + CN_0^{-\frac{3}{2}} M^{\frac{3}{2}} \\ &\lesssim \eta^{\frac{1}{5}} M^{\frac{3}{2}} \|P_{>M/4} u\|_{L_t^\infty L^2} + \eta^{-1} N_0^{-\frac{3}{2}} M^{\frac{3}{2}} \end{aligned} \quad (4.55)$$

The same estimate holds for $\|A'\|_{\dot{H}^{\frac{3}{2}}}$. Using again the fact that $e^{-it\sqrt{-\Delta}}v(t) \rightharpoonup 0$ weakly in $\dot{H}^{3/2}$ as $t \rightarrow T_+, T_-$, we estimate $\langle A, A' + B' \rangle_{\dot{H}^{\frac{3}{2}}}$ and $\langle A + B, A' \rangle_{\dot{H}^{\frac{3}{2}}}$ by

$$\begin{aligned} \left| \langle A, A' + B' \rangle_{\dot{H}^{\frac{3}{2}}} \right| &\lesssim \|A\|_{\dot{H}^{\frac{3}{2}}} M^{\frac{3}{2}} \|P_M v\|_{L_t^\infty L^2} \\ &\lesssim \left(\eta^{\frac{1}{5}} M^{\frac{3}{2}} \|P_{>M/4} u\|_{L_t^\infty L^2} + \frac{M^{\frac{3}{2}}}{\eta N_0^{\frac{3}{2}}} \right) M^{\frac{3}{2}} \|P_M v\|_{L_t^\infty L^2} \\ \left| \langle A + B, A' \rangle_{\dot{H}^{\frac{3}{2}}} \right| &\lesssim \left(\eta^{\frac{1}{5}} M^{\frac{3}{2}} \|P_{>M/4} u\|_{L_t^\infty L^2} + \frac{M^{\frac{3}{2}}}{\eta N_0^{\frac{3}{2}}} \right) M^{\frac{3}{2}} \|P_M v\|_{L_t^\infty L^2} \end{aligned} \quad (4.56)$$

Finally, we use the sharp Huygens principle to deduce that $\langle B, B' \rangle_{\dot{H}^{\frac{3}{2}}} = 0$. Indeed, we have

$$\begin{aligned} \langle B, B' \rangle_{\dot{H}^{\frac{3}{2}}} &= \int_{T_-}^{-\frac{C}{M}} \int_{\frac{C}{M}}^{T_+} \left\langle \chi\left(\frac{x}{c|t|}\right) P_M u^3(t), \nabla e^{-i(t-\tau)\sqrt{-\Delta}} \chi\left(\frac{\cdot}{c|\tau|}\right) P_M u^3(\tau) \right\rangle_{L^2} dt d\tau \\ &= 0 \end{aligned} \quad (4.57)$$

With say $c = \frac{1}{4}$, it follows from the sharp Huygens principle that the two terms inside the L^2 bracket above have disjoint spacial supports – the term on the left part of the bracket is supported in $|x| \leq \frac{1}{4}|t|$ and the term on the right side of the bracket is supported on $|x| > \frac{3}{4}|t - \tau| > \frac{3}{4}|t|$.

Now, we let $M = 2^k$ for $k < 0$. By (4.55), (4.56), and (4.57), we have

$$\begin{aligned} a_k^2 &\lesssim \left(\eta^{\frac{1}{5}} \sum_{j \geq k-3} 2^{\frac{3}{2}(k-j)} a_j + \eta^{-1} N_0^{-\frac{3}{2}} 2^{\frac{3}{2}k} \right)^2 \\ &\quad + a_k \left(\eta^{\frac{1}{5}} \sum_{j \geq k-3} 2^{\frac{3}{2}(k-j)} a_j + \eta^{-1} N_0^{-\frac{3}{2}} 2^{\frac{3}{2}k} \right) \end{aligned} \quad (4.58)$$

which implies that

$$a_k \lesssim \eta^{\frac{1}{5}} \sum_{j \geq k-3} 2^{\frac{3}{2}(k-j)} a_j + \eta^{-1} N_0^{-\frac{3}{2}} 2^{\frac{3}{2}k} \quad (4.59)$$

for all $k < 0$. For $k > 0$ we simply use the estimate $a_k \lesssim 1$. Recalling the definition of α_k we then have

$$\alpha_k \lesssim \eta^{\frac{1}{5}} \alpha_k + \sum_j 2^{-|j-k|} \inf(\eta^{-1} N_0^{-1} 2^{\frac{3}{2}j}, 1) \quad (4.60)$$

Now we fix $\eta > 0$ small enough in order to absorb the first term on the right into to the left-hand-side giving

$$\alpha_k \lesssim \sum_j 2^{-|j-k|} \inf(\eta^{-1} N_0^{-1} 2^{\frac{3}{2}j}, 1) \quad (4.61)$$

This implies that

$$\alpha_k \lesssim 2^k \quad \text{for } k < 0. \quad (4.62)$$

Since we have also set $\alpha_k := 1$ for $k \geq 0$ we have now proved that $\{2^{-\frac{3}{4}k} \alpha_k\} \in \ell^2$. This completes the proof of Proposition 4.5. \square

4.3. Higher regularity and additional decay for wave maps with the Compactness Property. In this section we extend the conclusions of Proposition 4.1 and Proposition 4.5 to the case when \vec{u} is a solution to either (1.18) or (1.22) with the Compactness Property on an interval I . In particular we prove the following results.

Proposition 4.6. *Suppose $\vec{u}(t)$ is a solution to either (1.18) or (1.22) with the Compactness Property on $I_{\max}(\vec{u})$ as in Theorem 3.3, i.e., assume that there exists a function $N : I_{\max} \rightarrow (0, \infty)$ so that the set*

$$K := \left\{ \left(\frac{1}{N(t)} u \left(t, \frac{\cdot}{N(t)} \right), \frac{1}{N^2(t)} u_t \left(t, \frac{\cdot}{N(t)} \right) \right) \mid t \in I \right\} \quad (4.63)$$

is pre-compact in $\dot{H}^{3/2} \times \dot{H}^{1/2}$. Then for all $t \in I_{\max}$,

$$\|\vec{u}(t)\|_{\dot{H}^{5/2} \times \dot{H}^{3/2}(\mathbb{R}^5)} \lesssim N(t).$$

with a constant that is uniform in $t \in I_{\max}$.

Proposition 4.7. *Let $\vec{u}(t)$ be a solution to either (1.18) or (1.22) with the Compactness Property as in Proposition 3.4, i.e., suppose that the scaling parameter $N(t)$ satisfies*

$$\inf_{t \in I_{\max}(\vec{u})} N(t) > 0$$

then for all $t \in I_{\max}$,

$$\|\vec{u}(t)\|_{\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}(\mathbb{R}^5)} \lesssim 1. \quad (4.64)$$

with a constant that is uniform in $t \in I_{\max}$.

The proof of Proposition 4.6 (respectively Proposition 4.7) is nearly identical to the proof of Proposition 4.1 (respectively Proposition 4.5) and we will thus omit many of the details. In fact, because of the assumption that $\vec{u}(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ with a uniform in $t \in I$ bound, we also have a uniform L_x^∞ estimate for ru , i.e., by Lemma 2.2 we have

$$\|ru\|_{L_t^\infty L_x^\infty} \lesssim \|u\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \lesssim 1$$

This means that for both $F_{\mathbb{S}^3}$ and $F_{\mathbb{H}^3}$ we have the estimate

$$|F_{\mathbb{S}^3}(r, u)|, |F_{\mathbb{H}^3}(r, u)| = |u^3 Z_{\mathbb{S}^3}(ru)|, |u^3 Z_{\mathbb{H}^3}(ru)| \lesssim |u|^3 \quad (4.65)$$

as was previously mentioned in (1.19) and (1.26). Therefore, we will only highlight below the instances in the proof of Proposition 4.6 in which the structure, rather than just the size of the nonlinearities comes into play. For the proof of Proposition 4.7 we provide even fewer details since the necessary additional techniques will already have been introduced in the proof of Proposition 4.6.

Sketch of the proof of Proposition 4.6. As we mentioned above, the proof follows the exact same argument as in the proof of Proposition 4.1 and therefore below we will only give a few details regarding the estimates where the structure, rather than just the size, of the nonlinearity enters into the argument. In particular, we will need replacements for the estimates in Claim 4.3, the related estimate (4.17), and Claim 4.4.

We begin by providing the estimates necessary to prove (4.25) which then implies (4.21) and thus the analog of Claim 4.4. Indeed here we can easily prove that

$$\|(1 - \chi)\left(\frac{x}{ct}\right)F(r, u)\|_{\dot{H}^{3/2}} \lesssim \|u\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \frac{1}{(ct)^2}.$$

where here $F(r, u)$ is either $F(u) = F_{\mathbb{S}^3}(r, u)$ or $F(r, u) = F_{\mathbb{H}^3}(r, u)$. We have

$$\|(1 - \chi)\left(\frac{r}{ct}\right)F(r, u)\|_{\dot{H}^{3/2}} \lesssim \|\nabla((1 - \chi)\left(\frac{r}{ct}\right)F(r, u))\|_{\dot{H}^{\frac{1}{2}}}$$

Since everything is radial above we compute, writing $F(r, u) = Z(ru)u^3$ with $Z = Z_{\mathbb{S}^3}$ or $Z = Z_{\mathbb{H}^3}$,

$$\begin{aligned} \partial_r \left((1 - \chi)\left(\frac{r}{ct}\right)Z(ru)u^3 \right) &= -\frac{1}{ct}\chi'\left(\frac{r}{ct}\right)Z(ru)u^3 + \\ &+ (1 - \chi\left(\frac{r}{ct}\right))(ru)Z'(ru) \left(\frac{u^3}{r} + u_r u^2 \right) + 2(1 - \chi\left(\frac{r}{ct}\right))Z(ru)u^2 u_r \end{aligned}$$

We estimate the right-hand-side above in $\dot{H}^{\frac{1}{2}}$ proceeding exactly as in the proof of Claim 4.4 using now the boundedness of ru and the structure of Z to note that $Z(ru)$ and $ruZ'(ru)$ are uniformly bounded. The only minor difference in the argument is that we note that Lemma 2.2 allows us to treat $\frac{u}{r}$ exactly as we would treat ∇u and indeed we have

$$\|r^{-1}u\|_{L^{\frac{5}{2}}} \lesssim \|u\|_{\dot{H}^{\frac{3}{2}}}, \quad \text{and} \quad \|r^{-1}u\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|u\|_{\dot{H}^{\frac{3}{2}}}$$

which are terms that arise after an application of the fractional product rule as in (4.23).

Next, we show how to proceed in the proofs of the analog Claim 4.3 and the related estimate (4.17). Setting $J := [-\delta/N(0), \delta/N(0)]$ and η as in Lemma 4.2 (of course now with a solution u to (1.18) or (1.22)) we know that

$$\|u\|_{L_t^2(J; L_x^{10})} \lesssim \eta \tag{4.66}$$

Examining the proof of Claim 4.3 we note that we have

$$a_k \lesssim a_k(0) + 2^{\frac{k}{2}} \|P_k F(r, u)\|_{L_t^1(J; L_x^2)} \tag{4.67}$$

where a_k and $a_k(0)$ are as in the statement of Claim 4.3 and \vec{u} is as in Proposition 4.6. As in the proof of Claim 4.3 we need to estimate $\|P_k F(r, u)\|_{L_t^1(J; L_x^2)}$ and

we claim that

$$2^{\frac{k}{2}} \|P_k F(r, u)\|_{L_t^1(J; L_x^2)} \lesssim \|u\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \eta^2 \sum_j 2^{-\frac{3}{2}|k-j|} a_j \quad (4.68)$$

To prove (4.68) we simply consider the power series expansion of $F(r, u)$. Indeed we have

$$F(r, u) = \sum_{n \geq 1} \frac{\iota^{2n+1}}{(2n+1)!} 2^{2n+1} r^{2n-2} u^{2n+1}$$

where $\iota = -1$ in the case that $F = F_{\mathbb{S}^3}$ and $\iota = 1$ in the case that $F = F_{\mathbb{H}^3}$. It follows that

$$\begin{aligned} 2^{\frac{k}{2}} \|P_k F(r, u)\|_{L_t^1(J; L_x^2)} &\lesssim 2^{\frac{k}{2}} \|P_k u^3\|_{L_t^1(J; L_x^2)} \\ &+ \sum_{n \geq 2} \frac{2^{2n+1}}{(2n+1)!} 2^{\frac{k}{2}} \|P_k(r^{2n-2} u^{2n+1})\|_{L_t^1(J; L_x^2)} \end{aligned} \quad (4.69)$$

From the proof of Claim 4.3 we know that

$$2^{\frac{k}{2}} \|P_k u^3\|_{L_t^1(J; L_x^2)} \lesssim \eta^2 \sum_{j > k-4} 2^{\frac{3}{2}(k-j)} a_j \quad (4.70)$$

Hence it suffices to show that

$$\sum_{n \geq 2} \frac{2^{2n+1}}{(2n+1)!} 2^{\frac{k}{2}} \|P_k(r^{2n-2} u^{2n+1})\|_{L_t^1(J; L_x^2)} \lesssim \|u\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \eta^2 \sum_j 2^{\frac{3}{2}|k-j|} a_j \quad (4.71)$$

To start, we note that

$$\begin{aligned} 2^{\frac{k}{2}} \|P_k(r^{2n-2} u^{2n+1})\|_{L_t^1(J; L_x^2)} &\lesssim 2^{\frac{k}{2}} \|P_k(r^{2n-2} u^{2n+1} - r^{2n-2} (P_{\leq k-4} u)^{2n+1})\|_{L_t^1(J; L_x^2)} \\ &+ 2^{\frac{k}{2}} \|P_k(r^{2n-2} (P_{\leq k-4} u)^{2n+1})\|_{L_t^1(J; L_x^2)} \end{aligned}$$

We estimate the first term on the right-hand-side above. We claim that

$$2^{\frac{k}{2}} \|P_k(r^{2n-2} u^{2n+1} - r^{2n-2} (P_{\leq k-4} u)^{2n+1})\|_{L_t^1(J; L_x^2)} \lesssim \|u\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \eta^2 \sum_{j > k-4} 2^{-\frac{3}{2}(k-j)} a_j \quad (4.72)$$

Indeed, writing $M = 2^k$, the left-hand-side can be broken into terms of the form

$$M^{\frac{1}{2}} \|P_M(r^{2n-2} u_{>M/4}^\ell u_{\leq M/4}^m)\|_{L^2} \quad (4.73)$$

with $\ell + m = 2n+1$ and $\ell \geq 1$ and we have introduced the notation $u_{\leq K} := P_{\leq K} u$. We then have, using Young's inequality and Lemma 2.2,

$$\begin{aligned} M^{\frac{1}{2}} \|P_M(r^{2n-2} u_{>M/4}^\ell u_{\leq M/4}^m)\|_{L^2} &\lesssim M^{\frac{1}{2}} \|\check{\phi}_M\|_{L^{\frac{5}{4}}} \|ru\|_{L^\infty}^{2n-2} \|u_{>M/4}^{\ell_2} u_{\leq M/4}^{r_2}\|_{L^{\frac{10}{7}}} \\ &\lesssim M^{\frac{3}{2}} \|u\|_{\dot{H}^{\frac{3}{2}}}^{2n-2} \|u_{>M/4}\|_{L^2} \|u\|_{L^{10}}^2 \end{aligned}$$

where $m_2 + r^2 = 3$ above and $m_2 \geq 1$. Integrating in time over the interval J proves (4.72). To finish the proof of (4.71) we show that

$$2^{\frac{k}{2}} \|P_k(r^{2n-2} (P_{\leq k-4} u)^{2n+1})\|_{L_t^1(J; L_x^2)} \lesssim \|u\|_{L_t^\infty \dot{H}^{\frac{3}{2}}} \eta^2 \sum_{j \leq k-4} 2^{\frac{3}{2}(j-k)} a_j \quad (4.74)$$

Again writing $M = 2^k$ we use Lemma 2.1 to deduce that

$$\begin{aligned}
M^{\frac{1}{2}} \|P_k(r^{2n-2}(P_{\leq k-4}u)^{2n+1})\|_{L^2} &\lesssim M^{-\frac{3}{2}} \|P_k \nabla^2(r^{2n-2}(P_{\leq k-4}u)^{2n+1})\|_{L^2} \\
&\lesssim M^{-\frac{3}{2}} \left((2n-2)(2n-3) \|P_k(r^{2n-4}u_{\leq M/4}^{2n+1})\|_{L^2} + \right. \\
&\quad + 2(2n-2)(2n+1) \|P_k(r^{2n-3}u_{\leq M/4}^{2n} \nabla u_{\leq M/4})\|_{L^2} + \\
&\quad + (2n+1)(2n) \|P_k(r^{2n-2}u_{\leq M/4}^{2n-1} (\nabla u_{\leq M/4})^2)\|_{L^2} + \\
&\quad \left. + (2n+1)(2n) \|P_k(r^{2n-2}u_{\leq M/4}^{2n-1} \nabla^2 u_{\leq M/4}^2)\|_{L^2} \right)
\end{aligned} \tag{4.75}$$

We estimate the first term inside the parentheses on the right-hand-side above as follows

$$\|P_k(r^{2n-4}u_{\leq M/4}^{2n+1})\|_{L^2} \lesssim \|ru\|_{L^\infty}^{2n-4} \|u_{\leq M/4}^5\|_{L^2}$$

Then note that replacing again $M = 2^k$ we have using Lemma 2.1 that

$$\begin{aligned}
\|(P_{\leq k-4}u)^5\|_{L^2} &\lesssim \sum_{j_1 \leq \dots \leq j_5 \leq k-4} \|P_{j_1}u\|_{L^{10}} \|P_{j_2}u\|_{L^{10}} \|P_{j_3}u\|_{L^\infty} \|P_{j_4}u\|_{L^\infty} \|P_{j_5}u\|_{L^{\frac{40}{3}}} \\
&\lesssim \|u\|_{L^{10}}^2 \sum_{j_3 \leq j_4 \leq j_5 \leq k-4} 2^{-\frac{1}{2}j_5} 2^{j_4} 2^{j_3} \|P_{j_3}u\|_{L^5} \|P_{j_4}u\|_{L^5} \|P_{j_5}u\|_{\dot{H}^{\frac{3}{2}}} \\
&\lesssim \|u\|_{L^{10}}^2 \|u\|_{\dot{H}^{\frac{3}{2}}}^2 \sum_{j \leq k-4} 2^{\frac{3}{2}j} a_j
\end{aligned}$$

The three remaining terms inside the parentheses on the right-hand-side of (4.75) are handled in an almost identical manner and we omit the details. Integrating in time over J and using (4.66) completes the proof of (4.74) and hence of (4.71).

We have now completed the proof of (4.68). By an analogous argument we can establish the analog of (4.17) for u as in Proposition 4.6. To complete the proof of Proposition 4.6 simply run the same argument as the proof of Proposition 4.1 with the estimates proved above inserted in the appropriate places. \square

Brief Sketch of the proof of Proposition 4.7. For the proof of Proposition 4.7 simply follow the exact same outline as the proof of Proposition 4.5 inserting arguments based on the power series expansions of $F_{\mathbb{S}^3}$ and $F_{\mathbb{H}^3}$ where necessary as in the sketch of Proposition 4.6 above. We omit the details. \square

5. RIGIDITY PART I: NONEXISTENCE OF COMPACT SELF-SIMILAR BLOW-UP

In this section we prove Proposition 3.5. We divide the argument into the following two subsections. The first deals with the case that the compact solution $\bar{u}(t)$ solves the focusing cubic equation (1.2). The second subsection deals with the case of self-similar wave maps with the compactness property. The arguments in both subsections are similar to and were inspired by the argument presented in [20, Section 4]. However, here the extra regularity gained in Section 4 is required to make the proof go through. In fact the cubic type nonlinearity would be a limiting case of the argument in [20] adapted to the present $5d$ setting and falls just out of the scope of the techniques used there. Thankfully, the extra regularity gained for critical elements in Section 4 is strong enough to allow us to adapt the argument from [20] to our situation, and in fact allows for a few simplifications.

5.1. Proof of Proposition 3.5 for solutions to the cubic equation (1.2). Suppose that \vec{u} is a self-similar solution to (1.2) with the compactness property as in Proposition 3.5. Without loss of generality we can assume, $T_+(\vec{u}) = 1$.

Lemma 5.1. *Let $\vec{u}(t)$ be as in Proposition 3.5. Then the set*

$$K_+ := \{((1-t)u(t, (1-t)r), (1-t)^2 u_t(t, (1-t)r) \mid t \in [0, 1)\}$$

is pre-compact in $(\dot{H}^2 \times \dot{H}^1) \cap (\dot{H}^{3/2} \times \dot{H}^{1/2})$.

Proof. This is immediate consequence of Proposition 4.1 as we can interpolate between the $\dot{H}^{5/2} \times \dot{H}^{3/2}$ bound and the $\dot{H}^{3/2} \times \dot{H}^{1/2}$ bound to control the $\dot{H}^2 \times \dot{H}^1$ norm of $((1-t)u(t, (1-t)\cdot), (1-t)^2 u_t(t, (1-t)\cdot))$. Compactness in $\dot{H}^{3/2} \times \dot{H}^{1/2}$ together with uniform boundedness in $\dot{H}^{5/2} \times \dot{H}^{3/2}$ then implies compactness in $\dot{H}^2 \times \dot{H}^1$. \square

Now we introduce new coordinates. Let $s = -\log(1-t)$, $e^{-s} = 1-t$, and $t = 1 - e^{-s}$. Then for $s \in [0, \infty)$

$$w(s, r) = e^{-s} u(1 - e^{-s}, e^{-s} r) \quad (5.1)$$

$$\partial_s w(s, r) = -w(s, r) - e^{-2s} r u_r(1 - e^{-s}, e^{-s} r) + e^{-2s} u_t(1 - e^{-s}, e^{-s} r). \quad (5.2)$$

Now, using [29, Lemma 4.15] we know that $T_+(\vec{u}) = 1$ implies that $\vec{u}(t)$ is supported in $B(0, 1-t)$, which means that we only need to consider $r \in [0, 1)$. The compact support then implies that the set K_+ in Lemma 5.1 is pre-compact in the inhomogeneous space $H^2 \times H^1(\mathbb{R}^5)$. Rephrasing this in terms of w , and using the compact support of $\vec{u}(t) \in B(0, 1-t)$ we see that the set

$$\tilde{K}_+ := \{w(s) \mid s \in [0, \infty)\} \quad (5.3)$$

is pre-compact in H^2 . Next, we derive an equation for $w(s)$. Given that $\vec{u}(t)$ solves (1.2) we have

$$\partial_s^2 w - \frac{(1-r^2)}{r^4} \partial_r(r^4 \partial_r w) + 2r \partial_r \partial_s w + 3 \partial_s w + 2w - w^3 = 0. \quad (5.4)$$

First we prove the following estimates using a monotonicity formula for an energy that we associate to (5.4).

Lemma 5.2. *Under the preceding assumptions*

$$\int_0^\infty \int_0^1 (1-r^2)^{-2} (\partial_s w(s, r))^2 r^4 dr < \infty.$$

Proof. By (5.4), we have

$$\begin{aligned} 0 &= \int r^4 (1-r^2)^{-1} w_s \left(w_{ss} - \frac{(1-r^2)}{r^4} \partial_r(r^4 w_r) + 2r w_{sr} + 3 \partial_s w + 2w - w^3 \right) dr \\ &= \frac{1}{2} \partial_s \int (\partial_s w)^2 (1-r^2)^{-1} r^4 dr + \frac{1}{2} \partial_s \int (\partial_r w)^2 r^4 dr + \int r^5 (1-r^2)^{-1} \partial_r (\partial_s w)^2 dr \\ &\quad + \partial_s \int w^2 (1-r^2)^{-1} r^4 dr - \frac{1}{4} \partial_s \int w^4 (1-r^2)^{-1} r^4 dr + 3 \int (\partial_s w)^2 (1-r^2)^{-1} r^4 dr. \end{aligned} \quad (5.5)$$

Defining

$$\begin{aligned} E(s) = & \frac{1}{2} \int_0^1 w_s^2(s)(1-r^2)^{-1} r^4 dr + \frac{1}{2} \int_0^1 w_r^2(s) r^4 dr \\ & + \int_0^1 w^2(s)(1-r^2)^{-1} r^4 dr - \frac{1}{4} \int_0^1 (1-r^2)^{-1} w^4(s) r^4 dr, \end{aligned} \quad (5.6)$$

and integrating (5.5) by parts, we have that

$$\frac{d}{ds} E(s) = 2 \int_0^1 (1-r^2)^{-2} w_s^2(s, r) r^4 dr. \quad (5.7)$$

Next, we observe that $E(0) \geq -C_0$ for some constant C_0 . Indeed, by inspection

$$E(0) \geq -\frac{1}{4} \int_0^1 (1-r^2)^{-1} w^4 r^4 dr. \quad (5.8)$$

Now $\|w\|_{H^2} \lesssim 1$, so by the radial Sobolev embedding this implies that $\|w_r\|_{L^\infty} \lesssim 1$ for $r \in [\frac{1}{2}, 1]$. Therefore, since $w(s)$ is supported on $B(0, 1)$, by the fundamental theorem of calculus $\|w(s, r)\|_{L^\infty} \lesssim (1-r)$ for $r \in [\frac{1}{2}, 1]$. Therefore, since $w(s)$ lies in a pre-compact subset of H^2 ,

$$\int_0^1 w^4(s)(1-r^2)^{-1} r^4 dr \leq C. \quad (5.9)$$

with a uniform in s constant. It remains to show that $E(s)$ is uniformly bounded above for all $s \in [0, \infty)$. Since $\|w(s, r)\|_{L^\infty} \lesssim 1-r$ for all $r \in [\frac{1}{2}, 1]$ and $w(s)$ lies in a pre-compact subset of H^2 , we can find a constant C independent of $s \in [0, \infty)$ so that

$$\int_0^1 w^2(s)(1-r^2)^{-1} r^4 dr \leq C.$$

Also since $\|w(s)\|_{\dot{H}^1}$ is uniformly bounded in $s \in [0, \infty)$,

$$\int_0^1 w_r^2(s) r^4 dr \leq C. \quad (5.10)$$

Also, by Hölder's inequality, the fundamental theorem of calculus, the support of w lying in $B(0, 1)$, and pre-compactness of \tilde{K}_+ in $H^2 \times H^1$ for all $t \in [0, 1]$ yields

$$|e^{-2s} r u_r(e^{-s} y, 1 - e^{-s})| + |e^{-2s} u_t(e^{-s} y, 1 - e^{-s})| \lesssim (1-r)^{1/2},$$

which combined with (5.2) and (5.10) proves that

$$\int_0^1 (1-r^2)^{-1} w_s^2(s, r) r^4 dr \leq C. \quad (5.11)$$

Then by the fundmantal theorem of calculus in s and (5.7), the proof of Lemma 5.2 is complete. \square

Next, we show that in fact $w(s)$ must converge to a *stationary solution* to (5.4) along a sequence $s_n \rightarrow \infty$. First note that since $(1-r^2)^{-2} \geq 1$, we can conclude from Lemma 5.2 that

$$\int_0^\infty \int_0^1 w_s^2(s, r) r^4 dr \leq C_0. \quad (5.12)$$

Now, let $s_n \rightarrow \infty$ be any sequence. Using the compactness of \tilde{K}_+ we can find $w^* \in H^2$ with $\text{supp } w^* \in B(0, 1)$ so that after passing to a subsequence we have

$$w(s_n) \rightarrow w^* \in H^2 \quad (5.13)$$

Next we claim that for any $T \geq 0$ we have

$$\|w(s_n + T) - w^*\|_{H^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (5.14)$$

Indeed, let $T \geq 0$. Then by the Fundamental Theorem of Calculus and (5.12) we deduce that

$$\begin{aligned} \|w(s_n + T) - w(s_n)\|_{L^2}^2 &= \int |w(s_n + T, r) - w(s_n, r)|^2 r^4 dr \\ &= \int \left| \int_{s_n}^{s_n+T} w_s(s, r) ds \right|^2 r^4 dr \\ &\leq T \int_{s_n}^{s_n+T} \int w_s^2(s, r) r^4 dr ds \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \quad (5.15)$$

This implies that for all $T \geq 0$ we have

$$\|w(s_n + T) - w^*\|_{L^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (5.16)$$

Another application of the compactness in H^2 of \tilde{K}_+ allows us to upgrade the above to obtain (5.14).

Next, define $\vec{v}_n(0) := (v_{n,0}(r), v_{n,1}(r))$ by

$$(v_{n,0}(r), v_{n,1}(r)) := (e^{-s_n} u(1 - e^{-s_n}, e^{-s_n} r), e^{-2s_n} u_t(1 - e^{-s_n}, e^{-s_n} r)) \quad (5.17)$$

By the pre-compactness of K_+ in $H^2 \times H^1$ we can find a strong limit $\vec{v}(0) = (v_0, v_1)$ so that

$$(v_{n,0}, v_{n,1}) \rightarrow \vec{v}(0) \in H^2 \times H^1$$

Let $\vec{v}_n(t)$ be the solution to (1.2) with initial data $\vec{v}_n(0)$ and let $\vec{v}(t)$ be the solution to (1.2) with data $\vec{v}(0)$. By the Perturbation Lemma 2.5 we know that for each $s \in [0, T_+(\vec{v})]$ we have

$$\|\vec{v}_n(s) - \vec{v}(s)\|_{\dot{H}^{3/2} \times \dot{H}^{1/2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (5.18)$$

Moreover, for $s < \min\{1, T_+(\vec{v})\}$ and T chosen so that $s = 1 - e^{-T}$ we have

$$v_n(s, r) = e^{-s_n} u(1 - e^{-s_n} + e^{-s_n} s, e^{-s_n} r) \quad (5.19)$$

Next observe that for T as above

$$w(s_n + T, r) = e^{-s_n} e^{-T} u(1 - e^{-s_n-T}, e^{-s_n-T} r) = e^{-T} v_n(1 - e^{-T}, e^{-T} r) \quad (5.20)$$

The left hand side above tends to w^* in H^2 so letting $n \rightarrow \infty$ above gives that for all T as above we have

$$w^*(r) = e^{-T} v(1 - e^{-T}, e^{-T} r) \quad (5.21)$$

Since \vec{v} is a solution to (1.2) we have that w^* is a solution to (5.4) which is *independent of s* . Therefore w^* solves

$$(1 - r^2)(w_{rr}^* + \frac{4}{r} w_r^*) + 2w^* - (w^*)^3 = 0. \quad (5.22)$$

Lemma 5.3. $w^* \equiv 0$.

Proof. Since $w^* \in H^2$, and $\text{supp } w^* \in B(0, 1)$ we have $\|w_{rr}^*\|_{L^2([\frac{1}{2}, 1])} \lesssim 1$. Since w^* is supported on $B(0, 1)$ and by the Fundamental Theorem of Calculus and Hölder's inequality,

$$|w_r^*(r)| \lesssim (1-r)^{1/2}, \quad \text{for } r \in [1/2, 1] \quad (5.23)$$

and from this we

$$|w^*(r)| \lesssim (1-r)^{3/2} \quad \text{for } r \in [1/2, 1] \quad (5.24)$$

Plugging (5.23) and (5.24) into (5.22), for $\frac{1}{2} \leq r \leq 1$, gives

$$|w_{rr}^*(r)| \leq \frac{4}{r} |w_r^*| + \frac{2}{1-r} |w^*| + \frac{|w^*|^3}{1-r} \leq C_0(1-r)^{1/2}. \quad (5.25)$$

Now also by (5.22), if $|w_{rr}^*| \leq C(1-r)^{1/2}$ for all $r \in [1-\delta, 1]$, and $C \leq C_0$, then by the Fundamental Theorem of Calculus, (5.23), and (5.24) (which implies that $w^*(1) = w_r^*(1) = 0$), we have

$$|((1-r^2)(w_{rr}^* + \frac{4}{r}w_r^*))| \leq 2|w^*(r)| + |w^*(r)|^3 \leq 2C(1-r)^{5/2} + C^3(1-r)^{15/2}. \quad (5.26)$$

Then since $1+r \geq 1$,

$$|w_{rr}^* + \frac{4}{r}w_r^*| \leq 2C(1-r)^{3/2} + C^3(1-r)^{13/2}. \quad (5.27)$$

For $r \in [\frac{1}{2}, 1]$, $|\frac{4}{r}w_r^*| \leq 8C(1-r)^{3/2}$. Therefore,

$$|w_{rr}^*(r)| \leq 10C(1-r)^{3/2} + 3C^3(1-r)^{13/2}. \quad (5.28)$$

Then for $r \in [1 - \frac{1}{100(1+C_0^3)}, 1]$, (5.28) implies that $|w_{rr}^*(r)| \leq \frac{C}{2}(1-r)^{1/2}$. Then by induction starting with (5.25), one can easily prove that $w_{rr}^*(r) = 0$ for $r \in [1 - \frac{1}{100(1+C_0^3)}, 1]$. Therefore, w^* is the solution to the elliptic partial differential equation on $B(0, 1-\delta)$, $\delta > 0$,

$$(1-|x|^2)\Delta w^* + 2w^* - (w^*)^3 = 0, \quad w^*|_{\partial B(0, 1-\delta)} = 0, \quad \partial_n w^*|_{\partial B(0, 1-\delta)} = 0. \quad (5.29)$$

Note that (5.29) is a *nondegenerate* elliptic equation with Dirichlet and Neumann boundary conditions. Therefore $w^* \equiv 0$. \square

At this point it is easy to conclude the proof. Since $w^* \equiv 0$ we can deduce from (5.21) that $\vec{v} \equiv 0$. But this means that $\vec{v}_n(0)$ converges to $(0, 0)$ in $H^2 \times H^1$. But this is impossible since we have assumed that $\vec{u}(t)$ is a blow-up solution. We have arrived at a contradiction and hence we have proved Proposition 3.5 for solutions to (1.2).

5.2. Nonexistence of self-similar wave maps with the compactness property. Next we exclude the possibility of compact, self-similar blow-up for solutions to (1.18) and (1.22). We again can assume that $T_+ = 1$ and that by [29, Lemma 4.15] we have $\text{supp } u(t, r), u_t(t, r) \in B(0, 1-t)$. Again we use this support property along with Proposition 4.6 to deduce that the set

$$K_+ := \{((1-t)u(t, (1-t)r), (1-t)^2 u_t(t, (1-t)r) \mid t \in [0, 1])\} \quad (5.30)$$

is in fact pre-compact in the inhomogeneous space $H^2 \times H^1$. A simple argument allows us to exclude this type of solution to the defocusing-type equation (1.22). Solutions to (1.22) have a positive definite conserved energy given by

$$\mathcal{E}_{\mathbb{H}^3}(\vec{u}) := \frac{1}{2} \int_0^\infty \left(u_r^2 + u_t^2 + 2 \frac{\sinh^2(ru) - (ru)^2}{r^4} \right) r^4 dr$$

Setting

$$\vec{v}(t, r) = (v(t, r), v_t(t, r)) := ((1-t)u(t, (1-t)r), (1-t)^2 u_t(t, (1-t)r)) \quad (5.31)$$

we have that $\vec{v}(t)$ is pre-compact in $H^2 \times H^1$ for $t \in [0, 1)$ and thus

$$\mathcal{E}_{\mathbb{H}^3}(\vec{u}(t)) \lesssim (1-t) (\|v_t(t)\|_{L^2}^2 + \|v_r(t)\|_{L^2}^2 + \|v(t)\|_{L^4}^4) \rightarrow 0 \text{ as } t \rightarrow 1 \quad (5.32)$$

Therefore $\mathcal{E}_{\mathbb{H}^3}(\vec{u}) = 0$ and it follows that $\vec{u} = (0, 0)$, which is contradiction since we assuming that $\vec{u}(t)$ blows up at $t = 1$.

In the case that $\vec{u}(t)$ solves the focusing-type equation (1.18) we follow the exact same argument as in the Section 5.1, defining $s = -\log(1-t)$, $e^{-s} = 1-t$, and $t = 1 - e^{-s}$ and for $s \in [0, \infty)$, setting

$$\begin{aligned} w(s, r) &= e^{-s} u(1 - e^{-s}, e^{-s} r) \\ \partial_s w(s, r) &= -w(s, r) - e^{-2s} r u_r(1 - e^{-s}, e^{-s} r) + e^{-2s} u_t(1 - e^{-s}, e^{-s} r). \end{aligned}$$

Of course in the present situation the nonlinearity $F_{\mathbb{S}^3}(r, u)$ is different than u^3 , however an inspection of the proof in Section 5.1 reveals that only the rough size of the nonlinearity

$$|F(r, u)| \lesssim |u|^3 \quad (5.33)$$

is needed along with the uniform estimate

$$(rw)^2 - \sin^2(rw) = ((rw) - \sin rw)(rw + \sin rw) \lesssim (rw)^4 \quad (5.34)$$

which arises when providing a lower bound for the analog of $E(0)$ as in (5.8). With these observations the proof proceeds exactly as in Section 5.1. We omit the details. This completes the proof of Proposition 3.5.

6. RIGIDITY PART II: PROOF OF PROPOSITION 3.4

In this section we prove Proposition 3.4. We use an argument developed in [25] that is based on exterior energy estimates for the free wave equation in \mathbb{R}^{1+5} . This method is a refinement of the channels of energy method developed by Duyckaerts, Kenig, and Merle in [19, 20]. In the current implementation, the argument closely resembles the one in [45, Section 5], however, we provide a detailed argument here for completeness. We begin with the following consequence of Proposition 4.5 and Proposition 4.7.

Lemma 6.1. *Let $\vec{u}(t)$ be a solution to either (1.2), (1.18), (1.22) with the Compactness Property on its maximal interval of existence $I_{\max}(\vec{u})$ as in Proposition 3.4, i.e., assume that the scaling parameter $N(t)$ satisfies*

$$\inf_{t \in I_{\max}} N(t) > 0.$$

Then, we have $\vec{u}(t) \in \dot{H}^1 \times L^2(\mathbb{R}^5)$ for all $t \in I_{\max}$, and in fact the trajectory

$$K_1 := \{\vec{u}(t) \mid t \in I_{\max}\} \quad (6.1)$$

is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^5)$. As a consequence, we have the following vanishing: Let $R > 0$ be arbitrary. Then

$$\limsup_{t \rightarrow T_-} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} = \limsup_{t \rightarrow T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} = 0 \quad (6.2)$$

where

$$\|\vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)}^2 := \int_{R+|t|}^{\infty} (u_t^2(t, r) + u_r^2(t, r)) r^4 dr$$

Proof. We begin by proving that K_1 is pre-compact in the energy space. We show that for every sequence $t_n \rightarrow T_+$, or $t_n \rightarrow T_-$ the sequence $\vec{u}(t_n)$ has a convergent subsequence in $\dot{H}^1 \times L^2$. Let $t_n \rightarrow T_+$. First assume that $N(t_n)$ remains bounded for all n . In this case we can, without loss of generality assume $N(t_n) = 1$ for all n and we have using Proposition 4.5 and interpolation

$$\begin{aligned} \|\vec{u}(t_n) - \vec{u}(t_m)\|_{\dot{H}^1 \times L^2} &\leq \|\vec{u}(t_n) - \vec{u}(t_m)\|_{\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}}^{\frac{2}{3}} \|\vec{u}(t_n) - \vec{u}(t_m)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}}^{\frac{1}{3}} \\ &\leq C \|\vec{u}(t_n) - \vec{u}(t_m)\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}}^{\frac{1}{3}} \end{aligned} \quad (6.3)$$

The claim then follows from the compactness in $\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ since we know that $\vec{u}(t_n)$ has a convergent subsequence there. Next assume that t_n has a subsequence, still denoted by t_n so that $N(t_n) \rightarrow \infty$. In this case we claim that $\vec{u}(t_n) \rightarrow 0$ in $\dot{H}^1 \times L^2$. To see this let η be a small number and find $c(\eta)$ as in Remark 4 so that

$$\int_{|\xi| \leq c(\eta)N(t)} |\xi|^3 |\hat{u}(t, \xi)|^2 d\xi \leq \eta_0, \quad (6.4)$$

Then

$$\begin{aligned} \|u(t_n)\|_{\dot{H}^1}^2 &= \int_{|\xi| \leq c(\eta)N(t_n)} |\xi|^2 |\hat{u}(t_n, \xi)|^2 d\xi + \int_{|\xi| \geq c(\eta)N(t_n)} |\xi|^2 |\hat{u}(t_n, \xi)|^2 d\xi \\ &\leq \int_{|\xi| \leq c(\eta)N(t_n)} |\xi|^2 |\hat{u}(t_n, \xi)|^2 d\xi + N(t_n)^{-1} c(\eta)^{-1} \|u(t_n)\|_{\dot{H}^{\frac{3}{2}}}^2 \end{aligned}$$

The second term on the right-hand-side above tends to zero as $n \rightarrow \infty$ since we have assumed $N(t_n) \rightarrow \infty$ as $n \rightarrow \infty$. For the first term, we interpolate

$$\begin{aligned} &\int_{|\xi| \leq c(\eta)N(t_n)} |\xi|^2 |\hat{u}(t_n, \xi)|^2 d\xi \leq \\ &\leq \left(\int_{|\xi| \leq c(\eta)N(t_n)} |\xi|^{\frac{3}{2}} |\hat{u}(t_n, \xi)|^2 d\xi \right)^{2/3} \left(\int_{|\xi| \leq c(\eta)N(t_n)} |\xi|^3 |\hat{u}(t_n, \xi)|^2 d\xi \right)^{1/3} \\ &\leq \eta \|u(t_n)\|_{\dot{H}^{3/4}} \lesssim \eta \end{aligned}$$

where the last line follows from Proposition 4.5 and Remark 4. The same argument works for the time derivatives $u_t(t_n)$. As η can be chosen arbitrarily small we have proved that $\vec{u}(t_n) \rightarrow 0$ in $\dot{H}^1 \times L^2$. Hence K_1 is pre-compact in the energy space.

Next we prove (6.2). First assume that $T_+ = \infty$. Then since K_1 is pre-compact in $\dot{H}^1 \times L^2$ it follows that for every $\varepsilon > 0$ there exists an $R(\varepsilon) > 0$ so that for all $t \in [0, \infty)$, we have

$$\|\vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R(\varepsilon))} \leq \varepsilon,$$

and (6.2) is a direct consequence of the above.

Next, assume that $T_+ < \infty$. A standard argument using compactness, see Remark 5(2), gives that in this case we must have

$$\text{supp } \vec{u}(t, \cdot) \subset B(0, T_+ - t) \quad (6.5)$$

where $B(0, r)$ denotes the ball of radius $r > 0$ in \mathbb{R}^5 . Then for any $R > 0$, we can find a $t_0 > 0$, $t_0 = t_0(R)$ so that

$$R + t > T_+ - t, \quad \forall t_0 \leq t < T_+$$

Hence,

$$\|\vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} = 0 \quad \forall t_0 \leq t < T_+.$$

The statements in (6.2) in the negative time directions follow from identical arguments. \square

6.1. Singular stationary solutions. Next, we construct one parameter families of stationary solutions to (1.2), (1.18) and (1.22) that do not lie in the critical space $\dot{H}^{\frac{3}{2}}(\mathbb{R}^5)$. We will later show that any nonzero pre-compact trajectory as in Proposition 3.4 has to be equal to one of these singular stationary solutions, which will yield a contradiction. We begin with the construction of a family of infinite energy stationary wave maps to \mathbb{S}^3 and to \mathbb{H}^3 .

6.1.1. Singular stationary wave maps. We prove the following result via a simple phase portrait analysis after a reduction to an autonomous ODE.

Lemma 6.2. *For any $\ell \in \mathbb{R}$, there exists a unique radial C^2 solution φ_ℓ^1 of*

$$-\varphi_{rr} - \frac{4}{r}\varphi_r = Z_{\mathbb{S}^3}(r\varphi)\varphi^3 \quad r > 0 \quad (6.6)$$

as well as a unique radial C^2 solution φ_ℓ^2 of

$$-\varphi_{rr} - \frac{4}{r}\varphi_r = Z_{\mathbb{H}^3}(r\varphi)\varphi^3 \quad r > 0 \quad (6.7)$$

where in both cases the solution has the asymptotic behavior

$$r^3\varphi_\ell^j(r) = \ell + O(r^{-4}) \quad \text{as } r \rightarrow \infty, \quad j = 1, 2 \quad (6.8)$$

The $O(\cdot)$ is determined by ℓ and vanishes for $\ell = 0$. Moreover, in each case $j = 1, 2$ for $\ell \neq 0$ we have $\varphi_\ell^j \notin L^5(\mathbb{R}^5)$, which means that $\varphi_\ell^j \notin \dot{H}^{\frac{3}{2}}(\mathbb{R}^5)$ by Sobolev embedding.

Proof. We first prove the lemma for the \mathbb{S}^3 target, which concerns solutions to (6.6). Recall then if we define $Q(r) := r\varphi(r)$ then φ solves (6.6), if and only if Q solves

$$Q_{rr} + \frac{2}{r}Q_r = \sin 2Q \quad (6.9)$$

Moreover, if we would like φ to satisfy (6.8) with $\ell \in \mathbb{R}$ then we need to impose the boundary condition

$$\lim_{r \rightarrow \infty} Q(r) = 0 \quad (6.10)$$

A standard trick is to introduce new variables $s = \log(r)$ and $\phi(s) = Q(r)$, and we obtain an autonomous differential equation for ϕ ,

$$\ddot{\phi} + \dot{\phi} = \sin(2\phi) \quad (6.11)$$

Note that the above is the equation for a damped pendulum. The proof thus reduces to an analysis of the phase portrait associated to (6.11). Setting $x(s) = \phi(s)$, $y(s) = \dot{\phi}(s)$ we can rewrite (6.11) as the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + \sin(2x) \end{pmatrix} =: X(x, y) \quad (6.12)$$

and we denote by Φ_s the flow associated to the vector field X . The equilibria of (6.12) occur at points $z_{k/2} = (\frac{k\pi}{2}, 0)$ where $k \in \mathbb{Z}$. For each $\frac{k}{2} = n \in \mathbb{Z}$ the flow has a saddle structure with eigenvalues $\lambda_+ = 1$, $\lambda_- = -2$, and the corresponding unstable and stable invariant subspaces for the linearized flow are given by the spans of $(1, \lambda_+) = (1, 1)$, respectively $(1, \lambda_-) = (1, -2)$. In a neighborhood $V \ni v_n = (n\pi, 0)$ we then have the one-dimensional invariant unstable manifold

$$W_n^u = \{(x, y) \in V \mid \Phi_s(x, y) \rightarrow v_n \text{ as } s \rightarrow -\infty\}$$

and the one-dimensional invariant stable manifold

$$W_n^s = \{(x, y) \in V \mid \Phi_s(x, y) \rightarrow v_n \text{ as } s \rightarrow \infty\}$$

which are tangent at z_n to the invariant subspaces of the linearized flow. In particular, for $n = 0$ we can parameterize the stable manifold W_0^s by

$$\phi_{0,\ell}(s) = \ell e^{-2s} + O(e^{-6s})$$

with $\ell \in \mathbb{R}$ determining all the coefficients of higher order and the $O(\cdot)$ vanishing for $\ell = 0$. It is straightforward to show that if $\ell > 0$ then $\phi_{0,\ell}(s)$ lies on the branch of the stable manifold that stays above 0 for all $s \in \mathbb{R}$, i.e., $\phi_{0,\ell}(s) > 0$ for all $s \in \mathbb{R}$. If $\ell = 0$ then $\phi_{0,\ell}(s) = 0$ for all $s \in \mathbb{R}$. Lastly, if $\ell < 0$ then $\phi_{0,\ell}(s) < 0$ for all $s \in \mathbb{R}$. Different choices of ℓ correspond to translations in s along the respective branches of the stable manifold, which is what uniqueness means in the statement of Lemma 6.2.

Multiplying the equation (6.11) by $\dot{\phi}$ and integrating from s to ∞ , one obtains the energy identity

$$-\dot{\phi}_{0,\ell}^2(s) + 2 \int_s^\infty \dot{\phi}_{0,\ell}^2(\rho) d\rho = -\sin^2(\phi_{0,\ell}(s)) \quad (6.13)$$

from which one deduces that if $\ell \neq 0$ it is impossible for $(\phi_{0,\ell}(s), \dot{\phi}_{0,\ell}(s))$ to ever equal $(0, 0)$ for any $s \in [-\infty, \infty)$. Passing back to the original variables, $r, Q(r)$ we have three trajectories

$$Q_{\ell\pm}(r) := \ell_{\pm} r^{-2} + O(r^{-6}), \quad Q_0(r) := 0 \quad (6.14)$$

To pass back to the $5d$ setting we set $\phi_\ell(r) := Q_\ell(r)/r$ proving (6.8) for our solutions to (6.6). When $\ell \neq 0$ we note that the fact that $\lim_{r \rightarrow 0} Q_\ell(r) \neq 0$ means that $\phi_\ell \notin L^5(\mathbb{R}^5)$ for $\ell \neq 0$.

Next we prove the lemma for solutions to (6.7). The proof is nearly identical so we only briefly summarize the argument. Again, first defining $Q(r) := r\phi(r)$ and then setting $s = \log(r)$ and $\phi(s) = Q(r)$, we can reduce matters to a phase portrait analysis for the autonomous ODE,

$$\ddot{\phi} + \dot{\phi} = \sinh(2\phi) \quad (6.15)$$

We note that here there is only one fixed point for the vector field $X(x, y) = (y, -y + \sinh 2x)$ at $(x, y) = (0, 0)$. The rest of the proof is identical to the \mathbb{S}^3 target case.

□

6.1.2. *Singular stationary solutions to (1.2).* We prove the an analogous result for the cubic equation, (1.2). Again below we reduce matters to a phase portrait analysis after a reduction to an autonomous ODE. For a alternative, but also simple approach to the analogous result for the focusing supercritical semilinear equation in 3d we refer the reader to [20, Proposition 3.2].

Lemma 6.3. *For any $\ell \in \mathbb{R}$, there exists a radial C^2 solution φ_ℓ of*

$$-\varphi_{rr} - \frac{4}{r}\varphi_r = \varphi^3 \quad r > 0 \quad (6.16)$$

with the asymptotic behavior

$$r^3\varphi_\ell(r) = \ell + O(r^{-4}) \quad \text{as } r \rightarrow \infty \quad (6.17)$$

The $O(\cdot)$ is determined by ℓ and vanishes for $\ell = 0$. Moreover, for $\ell \neq 0$ we have $\varphi_\ell \notin L^5(\mathbb{R}^5)$, which means that $\varphi_\ell \notin \dot{H}^{\frac{3}{2}}(\mathbb{R}^5)$ by Sobolev embedding.

Proof. Motivated by the reduction to an autonomous system in the plane in the case of the wave maps equations, we seek a similar reduction for solutions to (6.16). Observe that φ solves (6.16) if and only if $w(r) := r\varphi(r)$ solves

$$-w_{rr} - \frac{2}{r}w_r + \frac{2w - w^3}{r^2} = 0 \quad (6.18)$$

In order for a solution φ to (6.16) to satisfy (6.17) we also require

$$\lim_{r \rightarrow \infty} w(r) = 0 \quad (6.19)$$

To find the reduction to an autonomous equation, we set $s := \log(r)$ and $\phi(s) = w(r)$ and obtain the following equation for ϕ :

$$\ddot{\phi} + \dot{\phi} - 2\phi + \phi^3 = 0, \quad \lim_{s \rightarrow \infty} \phi(s) = 0 \quad (6.20)$$

Again, we set $x(s) = \phi(s)$, $y(s) = \dot{\phi}(s)$ and rewrite (6.11) as the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -y + 2x - x^3 \end{pmatrix} =: X(x, y) \quad (6.21)$$

and we denote by Φ_s the flow associated to the vector field X . The equilibria of (6.21) occur at points $z_0 = (0, 0)$ and $z_\pm = (\pm 1, 0)$. At $z_0 = (0, 0)$ the flow has a *saddle* structure with eigenvalues $\lambda_+ = 1$, $\lambda_- = -2$, and the corresponding unstable and stable invariant subspaces for the linearized flow are given by the spans of $(1, \lambda_+) = (1, 1)$, respectively $(1, \lambda_-) = (1, -2)$. The flow has a *sink* at each of the other two equilibrium points z_\pm . In a neighborhood $V \ni z_0 = (0, 0)$ we then have the one-dimensional invariant unstable manifold

$$W^u = \{(x, y) \in V \mid \Phi_s(x, y) \rightarrow z_0 \text{ as } s \rightarrow -\infty\}$$

and the one-dimensional invariant stable manifold

$$W^s = \{(x, y) \in V \mid \Phi_s(x, y) \rightarrow z_0 \text{ as } s \rightarrow \infty\}$$

which are tangent at $z_0 = (0, 0)$ to the invariant subspaces of the linearized flow. In particular, we can parameterize the stable manifold W^s by

$$\phi_\ell(s) = \ell e^{-2s} + O(e^{-6s})$$

with $\ell \in \mathbb{R}$ determining all the coefficients of higher order and $O(\cdot)$ vanishing for $\ell = 0$. Next, multiply the equation (6.20) by ϕ and integrate from s to $+\infty$ to obtain the energy identity

$$\int_s^\infty \dot{\phi}_\ell(\rho) d\rho = \frac{1}{2} \dot{\phi}_\ell^2(s) - \phi_\ell^2(s) + \frac{1}{4} \phi_\ell^4(s) \quad (6.22)$$

From the above it is clear that if $\ell \neq 0$ then it is impossible to have $(\phi_\ell(s), \dot{\phi}_\ell(s)) = (0, 0)$ for any $s \in [-\infty, \infty)$ since the right-hand-side would vanish while the left-hand-side would be strictly positive. Changing back to the original variables we have $w_\ell(r) = \phi_\ell(s)$ and

$$w_\ell(r) = \ell r^{-2} + O(r^{-6}) \quad \text{as } r \rightarrow \infty \quad (6.23)$$

and for $\ell \neq 0$ we have $\lim_{r \rightarrow 0} w(r) \neq 0$. Hence, setting $\varphi_\ell(r) = w(r)/r$ we see that $\varphi_\ell \notin L^5(\mathbb{R}^5)$. This finishes the proof. \square

6.2. Proof of Proposition 3.4. We are now ready to begin the proof of Proposition 3.4 in earnest. The proof proceeds in several steps and is similar to the arguments presented in [25, Section 5] as well as [45, Section 5]. The method is inspired by the “channels of energy” technique introduced in the seminal papers [19, 20]. The key ingredient in the proof are the following exterior energy estimates for the free radial wave equation in \mathbb{R}^{1+5} that were proved in [25, Section 4].

Proposition 6.4. [25, Proposition 4.1] *Let $\square V = 0$ in $\mathbb{R}_{t,x}^{1+5}$ with radial data $(V_0, V_1) \in \dot{H}^1 \times L^2(\mathbb{R}^5)$. Then with some absolute constant $c_0 > 0$ one has for every $R > 0$*

$$\max_{\pm} \limsup_{t \rightarrow \pm\infty} \int_{r > R+|t|}^\infty (V_t^2 + V_r^2)(t, r) r^4 dr \geq c_0 \|\pi_a^\perp(V_0, V_1)\|_{\dot{H}^1 \times L^2(r > R)}^2 \quad (6.24)$$

where $\pi_R = \text{Id} - \pi_R^\perp$ is the orthogonal projection onto the plane

$$P(R) := \{(c_1 r^{-3}, c_2 r^{-3}) \mid c_1, c_2 \in \mathbb{R}\}$$

in the space $\dot{H}^1 \times L^2(r > R)$. The left-hand side of (6.24) is zero for all data in this plane.

Remark 8. The presence of the projections π_R^\perp on the right-hand-side of (6.24) can be attributed to the fact that r^{-3} is the Newton potential in \mathbb{R}^5 . To see this, consider initial data $(V_0, 0) \in \dot{H}^1 \times L^2(r \geq R)$ which satisfies $(V_0, 0) = (r^{-3}, 0)$ for all $r \in \{r \geq R > 0\}$, with $V_0(r) \equiv 0$ on $\{r \leq R/2\}$. Then, using the finite speed of propagation, the corresponding free evolution $V(t, r)$ is given by $V(t, r) = r^{-3}$ on the region $\{r \geq R + |t|\}$. It is clear that the left-hand-side of (6.24) vanishes for this solution as $t \rightarrow \pm\infty$, and therefore an estimate without the projection is false. The other family of counterexamples to an estimate without the projection is generated by taking initial data $(0, V_1) = (0, r^{-3})$ on the exterior region $\{r \geq R > 0\}$ which has solution $V(t, r) = tr^{-3}$ on $\{r \geq R + |t|\}$.

Remark 9. The orthogonal projections π_R, π_R^\perp are precisely

$$\begin{aligned} \pi_R(V_0, 0) &= R^3 r^{-3} V(R), & \pi_R(0, V_1) &= R r^{-3} \int_R^\infty V_1(\rho) \rho d\rho, \\ \pi_R^\perp(V_0, 0) &= V_0(r) - R^3 r^{-3} V_0(a), & \pi_R^\perp(0, V_1) &= V_1(r) - R r^{-3} \int_R^\infty V_1(\rho) \rho d\rho, \end{aligned}$$

and thus we have

$$\begin{aligned}\|\pi_R(f, g)\|_{\dot{H}^1 \times L^2(r > R)}^2 &= 3R^3 f^2(R) + R \left(\int_R^\infty r g(r) dr \right)^2 \\ \|\pi_R^\perp(f, g)\|_{\dot{H}^1 \times L^2(r > R)}^2 &= \int_R^\infty f_r^2(r) r^4 dr - 3R^3 f^2(R) \\ &\quad + \int_R^\infty g^2(r) r^4 dr - R \left(\int_R^\infty r g(r) dr \right)^2.\end{aligned}$$

The idea behind the proof is that the exterior energy decay (6.2) together with the exterior energy estimates for the free equation, i.e., (6.24), can be combined to obtain exact asymptotics for the initial data of our pre-compact trajectory, $u_0(r) = u(0, r)$ and $u_1(r) = u_t(0, r)$ as $r \rightarrow \infty$, viz.,

$$\begin{aligned}r^3 u_0(r) &\rightarrow \ell_0 \quad \text{as } r \rightarrow \infty \\ r \int_r^\infty u_1(s) s ds &\rightarrow 0 \quad \text{as } r \rightarrow \infty\end{aligned}\tag{6.25}$$

We conclude by showing via a contradiction argument that $\vec{u}(t) = (0, 0)$ is the only solution to either (1.2), (1.18), or (1.22) with both the compactness property as in Proposition 3.4 and initial data with the above asymptotics. For this final portion of the proof, the exact structure of the nonlinearity in the underlying equation only enters at the level of the underlying elliptic theory, i.e., when we use the results of Lemma 6.3 for solutions to (1.2) and Lemma 6.2 when dealing with solutions to either (1.18) or (1.22). Given that we have already proved Lemma 6.3 and Lemma 6.2, and the fact that the conclusions of these results are identical, we carry out the rest of the argument under the general assumption that $\vec{u}(t)$ has the compactness property as in Proposition 3.4 and satisfies an equation of the form

$$\begin{aligned}u_{tt} - u_{rr} - \frac{4}{r} u_r &= F(r, u) \\ \vec{u}(0) &= (u_0, u_1)\end{aligned}\tag{6.26}$$

where

$$|F(r, u)| \lesssim |u|^3\tag{6.27}$$

We note that the estimate (6.27) is clear in the case of the nonlinearities in both (1.2) and (1.18). Moreover, since we will only be considering solutions $\vec{u}(t)$ to (1.22) with the compactness property, the bound on the $\dot{H}^{3/2} \times \dot{H}^{1/2}$ norm of $\vec{u}(t)$ implies L^∞ control over $ru(t, r)$, which means that in our situation the nonlinearity $F_{\mathbb{H}^3}(r, u) = Z_{\mathbb{H}^3}(ru)u^3$ also satisfies the estimate (6.27).

The proof now proceeds in several steps.

Step 1: First, we use the exterior energy estimates for the free radial wave equation in Proposition 6.4 together with (6.2) to deduce an inequality for a solutions $\vec{u}(t)$ with the compactness property as in Proposition 3.4. We remark that by using compactness this result holds uniformly in time.

Lemma 6.5. *Let $\vec{u}(t)$ be as in Lemma 6.1. There exists a number $R_0 > 0$ such that for all $R > R_0$ and for all $t \in I_{\max}(\vec{u})$ we have*

$$\|\pi_R^\perp \vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R)} \lesssim R^{-1} \|\pi_R \vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R)}^3\tag{6.28}$$

where π_R and π_R^\perp are as in Proposition 6.4. We remark that the constant in (6.28) is uniform in $t \in I_{\max}(\vec{u})$.

Before beginning the proof of Lemma 6.5 we need a preliminary result regarding a modified Cauchy problem with a small data theory in the energy space, which also captures the dynamics of the solution with the compactness property on exterior cones $\mathcal{C}_R := \{(t, r) \mid r \geq R + |t|\}$. In order to compare a nonlinear wave and the underlying free evolution with the same initial data in the energy space, we need to be in a small data setting where the Duhamel formula and Strichartz estimates can be combined effectively. That we only will consider the evolution on the exterior cone \mathcal{C}_R allows us to truncate the initial data as well as the nonlinearity in a way that renders the initial value problem *subcritical* relative to the energy, but still preserves the flow on \mathcal{C}_R .

We now fix a smooth radial function $\chi \in C^\infty(\mathbb{R}^5)$ where $\chi(|x|) = \chi(r) = 1$ for $\{r \geq 1\}$ and $\chi(r) = 0$ on $\{r \leq 1/2\}$. Then rescaling we set $\chi_R(r) := \chi(r/R)$ and for every $R > 0$ we consider the modified Cauchy problem:

$$\begin{aligned} h_{tt} - h_{rr} - \frac{4}{r}h_r &= F_R(r, h), \quad F_R(r, h) := \chi_R(r)F(r, u) \\ \vec{h}(0) &= (h_0, h_1) \end{aligned} \quad (6.29)$$

The point of this modification is that by forcing the nonlinearity to have support outside $B(0, R)$ we remove the super-critical nature of the problem. This allows for a small-data theory in the energy space $\dot{H}^1 \times L^2$ by way of Strichartz estimates and the usual contraction mapping based argument. We define a norm $Z(I)$ where $0 \in I \subset \mathbb{R}$ is a time interval by

$$\|\vec{h}\|_{Z(I)} = \|h\|_{L_t^2(I; L_x^5(\mathbb{R}^5))} + \|\vec{h}(t)\|_{L_t^\infty(I; \dot{H}^1 \times L^2(\mathbb{R}^5))} \quad (6.30)$$

Lemma 6.6. *There exists an ε_0 small enough, so that for all $R > 0$ and all initial data $\vec{h}(0) = (h_0, h_1) \in \dot{H}^1 \times L^2(\mathbb{R}^5)$ with*

$$\|\vec{h}(0)\|_{\dot{H}^1 \times L^2(\mathbb{R}^5)} < \varepsilon_0$$

there exists a unique global-in-time solution $\vec{h}(t) \in \dot{H}^1 \times L^2$ to (6.29). Moreover, $\vec{h}(t)$ satisfies

$$\|\vec{h}\|_{Z(\mathbb{R})} \lesssim \|\vec{h}(0)\|_{\dot{H}^1 \times L^2} \lesssim \varepsilon_0 \quad (6.31)$$

If we denote the free evolution of the same initial data by $\vec{h}_L(t) := S(t)\vec{h}(0)$, then we also have

$$\sup_{t \in \mathbb{R}} \|\vec{h}(t) - \vec{h}_L(t)\|_{\dot{H}^1 \times L^2} \lesssim R^{-1} \|\vec{h}(0)\|_{\dot{H}^1 \times L^2}^3 \quad (6.32)$$

Proof. As we mentioned above, the proof follows the usual argument based on Strichartz estimates and the Duhamel formula. Using Proposition 2.3 with $s = 1$, $(p, q, \gamma) = (2, 5, 1)$ and $(a, b, \rho) = (\infty, 2, 0)$, it suffices to control

$$\begin{aligned} \|\chi_R F(r, h)\|_{L_t^1 L_x^2} &\lesssim \|\chi_R |h|\|_{L_t^\infty L_x^{10}} \|h\|_{L_t^2 L_x^5}^2 \\ &\lesssim R^{-1} \|h\|_{L_t^\infty \dot{H}_x^1} \|h\|_{L_t^2 L_x^5}^2 \lesssim R^{-1} \|h\|_Z^3 \end{aligned} \quad (6.33)$$

where we have used (6.27) as well as radial Sobolev embedding, i.e., Lemma 2.2. \square

Remark 10. We remark that for every $t \in I_{\max}(u)$ the nonlinearity F_R in (6.29) satisfies

$$F_R(r, u) = F(r, u), \quad \forall r \geq R + |t|.$$

By the finite speed of propagation we can deduce that solutions to (6.29) and (6.26) agree on the exterior cone $\mathcal{C}_R := \{(t, r) \mid r \geq R + |t|\}$.

With the above remark in mind, we can now prove Lemma 6.5.

Proof of Lemma 6.5. As will always be the case in this section, $\vec{u}(t)$ is a solution with the compactness property as in Lemma 6.1. We will prove the inequality (6.28) first for time $t = 0$. The proof for all times $t \in I_{\max}$ for $R > R_0$ independent of t will follow immediately from the compactness property of $\vec{u}(t)$. First, define truncated data, $\vec{u}_R(0) = (u_{0,R}, u_{1,R})$ by

$$\begin{aligned} u_{0,R}(r) &:= \begin{cases} u_0(r) & \text{for } r \geq R \\ u_0(R) & \text{for } 0 \leq r \leq R \end{cases} \\ u_{1,R}(r) &:= \begin{cases} u_1(r) & \text{for } r \geq R \\ 0 & \text{for } 0 \leq r \leq R \end{cases} \end{aligned}$$

This new data is small in $\dot{H}^1 \times L^2$ for large $R > 0$. In fact,

$$\|\vec{u}_R(0)\|_{\dot{H}^1 \times L^2(\mathbb{R}^5)} = \|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)}$$

which means $R_0 > 0$ can be chosen large enough such that for all $R \geq R_0$ we have

$$\|\vec{u}_R(0)\|_{\dot{H}^1 \times L^2} \leq \delta \leq \min(\varepsilon_0, 1)$$

where ε_0 is chosen as in Lemma (6.6). By Lemma (6.6) we can find the unique global solution $\vec{u}_R(t)$ to (6.29) with initial data $\vec{u}_R(0)$, which satisfies (6.31) and (6.32).

Let $\vec{u}_{R,L}(t)$ be the free evolution of the initial data $\vec{u}_R(0)$. Note that

$$\begin{aligned} \|\vec{u}_R(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} &= \|\vec{u}_R(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} \\ &\geq \|u_{R,L}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} - \|\vec{u}_R(t) - u_{R,L}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} \end{aligned}$$

Now apply (6.32) to $\vec{u}_R(t)$ and take $R > R_0$ large enough so that

$$\begin{aligned} \|\vec{u}_R(t) - u_{R,L}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} &\leq \|\vec{u}_R(t) - u_{R,L}(t)\|_{\dot{H}^1 \times L^2} \\ &\lesssim R^{-1} \|\vec{u}_R(0)\|_{\dot{H}^1 \times L^2}^3 \\ &= R^{-1} \|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)}^3 \end{aligned}$$

Next, we put together the two preceding inequalities to obtain

$$\|\vec{u}_R(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} \geq \|u_{R,L}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} - CR^{-1} \|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)}^3. \quad (6.34)$$

We note that by Remark 10 we have

$$\vec{u}_R(t) = \vec{u}(t), \quad \forall (t, r) \in \mathcal{C}_R. \quad (6.35)$$

This means that we can use Lemma 6.1 to deduce that

$$\lim_{|t| \rightarrow \infty} \|\vec{u}_R(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} = 0 \quad (6.36)$$

Next we will let $t \rightarrow \pm\infty$ in (6.34) (the choice here is determined by the maximum in Proposition 6.4) and use Proposition 6.4 to provide a lower bound for the right-hand side of (6.34). Taking the limit as $t \rightarrow \pm\infty$, using (6.36), Proposition 6.4 and the fact that $\vec{u}_R(0) = \vec{u}(0)$ on $\{r \geq R\}$ we obtain that

$$\|\pi_R^\perp \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)} \lesssim R^{-1} \|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)}^3.$$

Now, use the orthogonality of the projection π_R to expand out the right-hand side above

$$\|\pi_R^\perp \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)} \lesssim R^{-1} \left(\|\pi_R \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)}^2 + \|\pi_R^\perp \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)}^2 \right)^{\frac{3}{2}}.$$

To complete the proof, we let R_0 be large enough so that we can absorb the π_R^\perp term above on the right-hand-side into the left-hand-side which gives

$$\|\pi_R^\perp \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)} \lesssim R^{-1} \|\pi_R \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq R)}^3,$$

proving the lemma for $t = 0$. To see that (6.28) holds for all $t \in \mathbb{R}$ we note that by the pre-compactness of K we can choose $R_0 = R_0(\varepsilon_0)$ so that for all $R \geq R_0$ we have

$$\|\vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R)} \leq \min(\varepsilon_0, 1)$$

uniformly in $t \in I_{\max}$. Now simply repeat the above argument with truncated initial data at time $t = t_0$ and $R \geq R_0$ given by

$$\begin{aligned} u_{0,R,t_0}(r) &:= \begin{cases} u(t_0, r) & \text{for } r \geq R \\ u_0(t_0, R) & \text{for } 0 \leq r \leq R \end{cases} \\ u_{1,R,t_0}(r) &:= \begin{cases} u_t(t_0, r) & \text{for } r \geq R \\ 0 & \text{for } 0 \leq r \leq R \end{cases} \end{aligned}$$

This ends the proof. \square

Step 2: Next, we will use the estimates in Lemma 6.5 to deduce the asymptotic behavior of $\vec{u}(0, r)$ as $r \rightarrow \infty$ that was described in (6.25). To be precise, we establish the following lemma.

Lemma 6.7. *Let $\vec{u}(t)$ be as in Proposition 3.4. Then we can find an $\ell_0 \in \mathbb{R}$ such that*

$$r^3 u_0(r) \rightarrow \ell_0 \quad \text{as } r \rightarrow \infty \tag{6.37}$$

$$r \int_r^\infty u_1(\rho) \rho d\rho \rightarrow 0 \quad \text{as } r \rightarrow \infty \tag{6.38}$$

at the rates

$$|r^3 u_0(r) - \ell_0| = O(r^{-4}) \quad \text{as } r \rightarrow \infty \tag{6.39}$$

$$\left| r \int_r^\infty u_1(\rho) \rho d\rho \right| = O(r^{-2}) \quad \text{as } r \rightarrow \infty \tag{6.40}$$

For the proof of Lemma 6.7 we make the following substitutions, which simplify the notation. Set

$$\begin{aligned} v_0(t, r) &:= r^3 u(t, r), \\ v_1(t, r) &:= r \int_r^\infty u_t(t, \rho) \rho d\rho. \end{aligned} \tag{6.41}$$

We will also often write $v_0(r) := v_0(0, r)$, $v_1(r) := v_1(0, r)$. Writing the projections in terms of v_0, v_1 gives

$$\begin{aligned} \|\pi_R \vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R)}^2 &= 3R^{-3} v_0^2(t, R) + R^{-1} v_1^2(t, R) \\ \|\pi_R^\perp \vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R)}^2 &= \int_R^\infty \left(\frac{1}{r} \partial_r v_0(t, r) \right)^2 dr + \int_R^\infty (\partial_r v_1(t, r))^2 dr, \end{aligned} \quad (6.42)$$

and we can rephrase the conclusions of Lemma 6.5 in terms of the newly defined functions (v_0, v_1) .

Lemma 6.8. *Let (v_0, v_1) be defined as in (6.41). Then there is an $R_0 > 0$ so that for all $R \geq R_0$,*

$$\left(\int_R^\infty \left(\frac{1}{r} \partial_r v_0(t, r) \right)^2 + (\partial_r v_1(t, r))^2 dr \right)^{\frac{1}{2}} \lesssim R^{-1} (3R^{-3} v_0^2(t, R) + R^{-1} v_1^2(t, R))^{\frac{3}{2}}$$

where the implicit constant above is uniform in $t \in I_{\max}(\vec{u})$.

Lemma 6.8 is now used to establish difference estimates. We let $\delta_1 > 0$ be small (to be determined precisely below) so that $\delta_1 \leq \varepsilon_0^2$ where ε_0 is the small constant in Lemma 6.6. We also choose $R_1 = R_1(\delta_1)$ large enough such that for all $R \geq R_1$,

$$\|\vec{u}(t)\|_{\dot{H}^1 \times L^2(r \geq R)}^2 \leq \delta_1 \leq \varepsilon_0^2, \quad R_1^{-1} \leq \delta_1 \quad (6.43)$$

Such an R_1 exists by the assumption that \vec{u} satisfies the compactness property on I_{\max} .

Corollary 6.9. *Let R_1 be as above. Then for all $R_1 \leq r \leq r' \leq 2r$ and for all $t \in \mathbb{R}$ we have*

$$|v_0(t, r) - v_0(t, r')| \lesssim r^{-4} |v_0(t, r)|^3 + r^{-1} |v_1(t, r)|^3 \quad (6.44)$$

$$|v_1(t, r) - v_1(t, r')| \lesssim r^{-5} |v_0(t, r)|^3 + r^{-2} |v_1(t, r)|^3 \quad (6.45)$$

with the above estimates holding uniformly in $t \in \mathbb{R}$.

For convenience we also record a rewording of Corollary 6.9 that is a direct consequence of (6.42).

Corollary 6.10. *Let R_1, δ_1 be defined as in (6.43). Then for all r, r' with $R_1 \leq r \leq r' \leq 2r$ and for all $t \in \mathbb{R}$ we have*

$$|v_0(t, r) - v_0(t, r')| \lesssim r^{-1} \delta_1 |v_0(t, r)| + \delta_1 |v_1(t, r)| \quad (6.46)$$

$$|v_1(t, r) - v_1(t, r')| \lesssim r^{-2} \delta_1 |v_0(t, r)| + r^{-1} \delta_1 |v_1(t, r)| \quad (6.47)$$

where all of the above hold uniformly in $t \in I_{\max}(\vec{u})$.

Proof of Corollary 6.9. This is an immediate consequence of Lemma 6.8. If $r \geq R_1$ and $r \leq r' \leq 2r$, then using Lemma 6.8 gives

$$\begin{aligned} |v_0(t, r) - v_0(t, r')| &\leq \left(\int_r^{r'} |\partial_r v_0(t, \rho)| d\rho \right) \\ &\leq \left(\int_r^{r'} \left| \frac{1}{\rho} \partial_r v_0(t, \rho) \right|^2 d\rho \right)^{\frac{1}{2}} \left(\int_r^{r'} \rho^2 d\rho \right)^{\frac{1}{2}} \\ &\lesssim r^{\frac{3}{2}} \left[r^{-1} (3r^{-3} v_0^2(t, r) + r^{-1} v_1^2(t, r))^{\frac{3}{2}} \right] \\ &\lesssim r^{-4} |v_0(t, r)|^3 + r^{-1} |v_1(t, r)|^3. \end{aligned}$$

In the same fashion

$$\begin{aligned} |v_1(t, r) - v_1(t, r')| &\leq \left(\int_r^{r'} |\partial_r v_1(t, \rho)|^2 d\rho \right)^{\frac{1}{2}} \left(\int_r^{r'} d\rho \right)^{\frac{1}{2}} \\ &\lesssim r^{-5} |v_0(t, r)|^3 + r^{-2} |v_1(t, r)|^3, \end{aligned}$$

which proves the difference estimates. \square

Now, we use the difference estimates to establish an upper bound on the growth rates of $v_0(t, r)$ and $v_1(t, r)$.

Claim 6.11. *Let $v_0(t, r)$ and $v_1(t, r)$ be as in (6.41). Then*

$$|v_0(t, r)| \lesssim r^{\frac{1}{6}} \quad (6.48)$$

$$|v_1(t, r)| \lesssim r^{\frac{1}{6}} \quad (6.49)$$

with constants that are uniform in $t \in I_{\max}$.

Proof. Again, it suffices to deduce the claim when $t = 0$ because the argument uses only the estimates in this section that hold uniformly in $t \in I_{\max}$.

Let $r_0 \geq R_1$ be fixed and note that by setting $r = 2^n r_0$, $r' = 2^{n+1} r_0$ in the difference estimates (6.46), (6.47) we have for all positive integers $n \in \mathbb{N}$,

$$|v_0(2^{n+1} r_0)| \leq (1 + C_1 (2^n r_0)^{-1} \delta_1) |v_0(2^n r_0)| + C_1 \delta_1 |v_1(2^n r_0)| \quad (6.50)$$

$$|v_1(2^{n+1} r_0)| \leq (1 + C_1 (2^n r_0)^{-1} \delta_1) |v_1(2^n r_0)| + C_1 \delta_1 (2^n r_0)^{-2} |v_0(2^n r_0)| \quad (6.51)$$

We introduce the notation

$$a_n := |v_0(2^n r_0)|$$

$$b_n := |v_1(2^n r_0)|$$

Adding (6.50) to (6.51) gives

$$\begin{aligned} a_{n+1} + b_{n+1} &\leq (1 + C_1 \delta_1 ((2^n r_0)^{-1} + (2^n r_0)^{-2})) a_n + (1 + C_1 \delta_1 (1 + (2^n r_0)^{-1})) b_n \\ &\leq (1 + 2C_1 \delta_1) (a_n + b_n) \end{aligned}$$

Arguing inductively we conclude that for all $n \in \mathbb{N}$,

$$(a_n + b_n) \leq (1 + 2C_1 \delta_1)^n (a_0 + b_0).$$

Now let δ_1 be small enough so that $(1 + 2C_1 \delta_1) \leq 2^{\frac{1}{6}}$, giving

$$\begin{aligned} a_n &\leq C (2^n r_0)^{\frac{1}{6}}, \\ b_n &\leq C (2^n r_0)^{\frac{1}{6}}. \end{aligned} \quad (6.52)$$

Note that $C = C(r_0)$ but this is irrelevant for our purposes since we have fixed r_0 . Observe that (6.52) proves (6.48) and (6.49) for $r = 2^n r_0$. The estimates (6.48) and (6.49) for arbitrary r now follow by combining (6.52) together with the difference estimates (6.44), (6.45). \square

We are now ready to begin extracting a limits. We first show that $v_1(t, r)$ tends to zero in order to get the correct decay rate for $v_0(t, r)$. This requires several steps.

Claim 6.12. *For every $t \in I_{\max}$ there is a number $\ell_1(t) \in \mathbb{R}$ so that*

$$|v_1(t, r) - \ell_1(t)| = O(r^{-2}) \quad \text{as } r \rightarrow \infty \quad (6.53)$$

where the implicit constant is uniform in $t \in I_{\max}$.

Proof. Again it suffices to prove the claim for $t = 0$. Let $r_0 \geq R_1$ with R_1 as in (6.43). Inserting (6.48), (6.49) into the difference estimate (6.45) yields

$$\begin{aligned} |v_1(2^{n+1}r_0) - v_1(2^n r_0)| &\lesssim (2^n r_0)^{-5} (2^n r_0)^{\frac{1}{2}} + (2^n r_0)^{-2} (2^n r_0)^{\frac{1}{2}} \\ &\lesssim (2^n r_0)^{-\frac{3}{2}} \end{aligned} \quad (6.54)$$

Therefore the infinite series

$$\sum_n |v_1(2^{n+1}r_0) - v_1(2^n r_0)| < \infty,$$

is bounded, which then implies that there exists $\ell_1 \in \mathbb{R}$ so that

$$\lim_{n \rightarrow \infty} v_1(2^n r_0) = \ell_1.$$

Another application of the difference estimates along with the growth estimates (6.48), (6.49) gives,

$$\lim_{r \rightarrow \infty} v_1(r) = \ell_1.$$

To establish the correct rate of convergence, we remark that the limit above implies that $|v_1(r)|$ is bounded, and therefore the same logic that give (6.54) can be used to get

$$|v_1(2^{n+1}r) - v_1(2^n r)| \lesssim (2^n r)^{-2}$$

for all r large enough. Finally,

$$|v_1(r) - \ell_1| = \left| \sum_{n \geq 0} (v_1(2^{n+1}r) - v_1(2^n r)) \right| \lesssim r^{-2} \sum_{n \geq 0} 2^{-2n} \lesssim r^{-2}$$

which finishes the argument. \square

Next, we show that $\ell_1(t) = \ell_1$ is independent of t .

Claim 6.13. *The function $\ell_1(t)$ in Claim 6.12 is independent of $t \in I_{\max}$, that is there is a fixed number $\ell_1 \in \mathbb{R}$ so that $\ell_1(t) = \ell_1$ for all $t \in I_{\max}$.*

Proof. Recall that

$$v_1(t, r) := r \int_r^\infty u_t(t, \rho) \rho d\rho$$

By Claim (6.12), we see that

$$\ell_1(t) = r \int_r^\infty u_t(t, \rho) \rho d\rho + O(r^{-2}) \quad \text{as } r \rightarrow \infty$$

Let $t_1, t_2 \in I_{\max}$ be arbitrary with, say $t_1 \neq t_2$. We prove that

$$\ell_1(t_2) - \ell_1(t_1) = 0 \quad (6.55)$$

Indeed, averaging in $R \geq R_1$ yields

$$\begin{aligned} \ell_1(t_2) - \ell_1(t_1) &= \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty (u_t(t_2, r) - u_t(t_1, r)) r dr \right) ds + O(R^{-2}) \\ &= \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty \int_{t_1}^{t_2} u_{tt}(t, r) dt r dr \right) ds + O(R^{-2}) \end{aligned}$$

Using the fact that $\vec{u}(t)$ is a solution to (6.26), we can rewrite the above integral as

$$\begin{aligned} &= \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty (ru_{rr}(t, r) + 4u_r(t, r)) dr \right) ds dt + \\ &+ \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty F(r, u) dr \right) ds dt + O(R^{-2}) \\ &= I + II + O(R^{-2}) \end{aligned}$$

To estimate I we integrate by parts twice:

$$\begin{aligned} I &= \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty \frac{1}{r^3} \partial_r(r^4 u_r(t, r)) dr \right) ds dt \\ &= 3 \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty u_r(t, r) dr \right) ds dt - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} s^2 u_r(t, s) ds dt \\ &= -3 \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r u(t, r) dr dt - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r^2 u_r(t, r) dr dt \\ &= - \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} r u(t, r) dr dt + \int_{t_1}^{t_2} (Ru(t, R) - 2Ru(t, 2R)) dt \end{aligned} \quad (6.56)$$

To bound the above we recall that by the definition of v_0 and (6.48) we have

$$r^3 |u(t, r)| := |v_0(t, r)| \lesssim r^{\frac{1}{6}} \quad (6.57)$$

uniformly in $t \in I_{\max}$, and hence $|u(t, r)| \lesssim r^{-\frac{17}{6}}$ uniformly in $t \in I_{\max}$. Inserting (6.57) into the last line in (6.56) gives,

$$I = |t_2 - t_1| O(R^{-\frac{11}{6}})$$

Next we estimate II . We again use (6.57) to see that for $r > R_1$ we have

$$|F(r, u(t, r))| \lesssim |u(t, r)|^3 \lesssim r^{-\frac{17}{2}} \lesssim r^{-8}$$

Therefore,

$$II \lesssim \int_{t_1}^{t_2} \frac{1}{R} \int_R^{2R} \left(s \int_s^\infty r^{-8} dr \right) ds dt = |t_2 - t_1| O(r^{-6})$$

Combining all of the above we have

$$|\ell_1(t_2) - \ell_1(t_1)| = O(R^{-\frac{11}{6}}) \quad \text{as } R \rightarrow \infty$$

which means that $\ell_1(t_1) = \ell_1(t_2)$. □

Next, we prove that $\ell_1 \equiv 0$.

Claim 6.14. $\ell_1 = 0$.

Proof. Suppose that $\ell_1 \neq 0$. We have shown that for $R \geq R_1$ and for every $t \in I_{\max}$,

$$R \int_R^\infty u_t(t, r) r dr = \ell_1 + O(R^{-2}),$$

Hence, by taking R large, the left-hand side will have the same sign as ℓ_1 , for all $t \in I_{\max}$. Therefore, we can choose $R \geq R_1$ large enough such that for all $t \in I_{\max}$,

$$\left| R \int_R^\infty u_t(t, r) r dr \right| \geq \frac{|\ell_1|}{2}. \quad (6.58)$$

We now consider two cases. First suppose that $T_+ < \infty$. Then, since both

$$\text{supp}(u), \text{supp } u_t \subset B(0, T_+ - t)$$

we can find t close enough to T_+ so that the left-hand-side of (6.58) is identically zero, which is a contradiction if $\ell_1 \neq 0$.

Next, consider the case that $T_+ = \infty$. By integrating (6.58) from $t = 0$ to $t = T$ we obtain

$$\left| \int_0^T R \int_R^\infty u_t(t, r) r dr dt \right| \geq T \frac{|\ell_1|}{2}.$$

However, integrating in t on the left-hand side and using (6.57) we also have

$$\begin{aligned} \left| R \int_R^\infty \int_0^T u_t(t, r) r dt dr \right| &= \left| R \int_R^\infty (u(T, r) - u(0, r)) r dr \right| \\ &\lesssim R \int_R^\infty r^{-\frac{11}{6}} dr \lesssim R^{\frac{1}{6}}. \end{aligned}$$

Thus for a large R fixed we have

$$T \frac{|\ell_1|}{2} \lesssim R^{\frac{1}{6}},$$

which is a contradiction once T is taken large enough. \square

We are ready to complete the proof of Lemma 6.7.

Proof of Lemma 6.7. First we remark that by putting together Claims 6.12, 6.13, and 6.14, we have proved (6.38) as well as (6.40), i.e.,

$$|v_1(r)| = O(r^{-2}) \quad \text{as } r \rightarrow 0. \quad (6.59)$$

It thus remains to prove that there exists $\ell_0 \in \mathbb{R}$ such that

$$|v_0(r) - \ell_0| = O(r^{-4}) \quad \text{as } r \rightarrow \infty.$$

To show the above, we plug (6.59) along with (6.48) into the difference estimate (6.44). We see that for fixed $r_0 \geq R_1$ and all $n \in \mathbb{N}$,

$$\begin{aligned} |v_0(2^{n+1}r_0) - v_0(2^n r_0)| &\lesssim (2^n r_0)^{-4} (2^n r_0)^{\frac{1}{2}} + (2^n r_0)^{-1} (2^n r_0)^{-6} \\ &\lesssim (2^n r_0)^{-\frac{7}{2}}. \end{aligned}$$

Thus the infinite series

$$\sum_n |v_0(2^{n+1}r_0) - v_0(2^n r_0)| < \infty,$$

which then implies that there exists $\ell_0 \in \mathbb{R}$ so that

$$\lim_{n \rightarrow \infty} v_0(2^n r_0) = \ell_0.$$

Another application of the difference estimates (6.44) and the fact that $|v_1(r)| \rightarrow 0$ gives that,

$$\lim_{r \rightarrow \infty} v_0(r) = \ell_0.$$

To prove the rate of convergence, we observe that $|v_0(r)|$ is bounded since it has a limit, and therefore the difference estimate can be upgraded show that

$$|v_0(2^{n+1}r) - v_0(2^n r)| \lesssim (2^n r)^{-4}$$

for every $r \geq R_1$. Thus,

$$|v_0(r) - \ell_0| = \left| \sum_{n \geq 0} (v_1(2^{n+1}r) - v_1(2^n r)) \right| \lesssim r^{-4} \sum_{n \geq 0} 2^{-4n} \lesssim r^{-2},$$

completing the proof. \square

Step 3: The final step in the proof of Proposition 3.4 is to conclude that $\vec{u}(t, r) \equiv (0, 0)$. We divide the final step into two cases depending on whether ℓ_0 in Lemma 6.7 is zero or nonzero.

Case 1: $\ell_0 = 0$ **implies** $\vec{u}(t) \equiv (0, 0)$:

We formulate the above as a lemma:

Lemma 6.15. *Let $\vec{u}(t)$ be as in Proposition 3.4 and let $\ell_0 \in \mathbb{R}$ be as in Lemma 6.7. If $\ell_0 = 0$ then $\vec{u}(t) \equiv (0, 0)$.*

To prove the lemma, we will first prove the following preliminary claim, which says that if $\ell_0 = 0$ then (u_0, u_1) is compactly supported. We conclude the proof of the lemma by showing that the only solution with pre-compact trajectory and compactly supported initial data is $\vec{u}(t) = (0, 0)$.

Claim 6.16. *Let ℓ_0 be as in Lemma 6.7. If $\ell_0 = 0$ then (u_0, u_1) is compactly supported.*

Proof. If $\ell_0 = 0$, then for $r \geq R_1$ we have

$$\begin{aligned} |v_0(r)| &= O(r^{-4}) \quad \text{as } r \rightarrow \infty \\ |v_1(r)| &= O(r^{-2}) \quad \text{as } r \rightarrow \infty \end{aligned} \tag{6.60}$$

Hence for $r_0 \geq R_1$,

$$|v_0(2^n r_0)| + |v_1(2^n r_0)| \lesssim (2^n r_0)^{-4} + (2^n r_0)^{-2} \tag{6.61}$$

On the other hand, the difference estimates (6.44) and (6.45) and (6.60) yield

$$\begin{aligned} |v_0(2^{n+1}r_0)| &\geq (1 - C_1(2^n r_0)^{-12}) |v_0(2^n r_0)| - C_1(2^n r_0)^{-5} |v_1(2^n r_0)| \\ |v_1(2^{n+1}r_0)| &\geq (1 - C_1(2^n r_0)^{-6}) |v_0(2^n r_0)| - C_1(2^n r_0)^{-13} |v_1(2^n r_0)| \end{aligned}$$

For large r_0 we combine the above two lines to deduce that

$$|v_0(2^{n+1}r_0)| + |v_1(2^{n+1}r_0)| \geq (1 - 2C_1 r_0^{-5})(|v_0(2^n r_0)| + |v_1(2^n r_0)|)$$

Now fix r_0 large enough so that $2C_1 r_0^{-5} < \frac{1}{4}$. By an inductive argument we conclude that

$$(|v_0(2^n r_0)| + |v_1(2^n r_0)|) \geq \left(\frac{3}{4}\right)^n (|v_0(r_0)| + |v_1(r_0)|)$$

Next, use (6.61) to estimate the left-hand-side above. We have

$$\left(\frac{3}{4}\right)^n (|v_0(r_0)| + |v_1(r_0)|) \lesssim 2^{-2n} r_0^{-2}$$

which yields

$$3^n (|v_0(r_0)| + |v_1(r_0)|) \lesssim 1,$$

However, this is impossible unless $(v_0(r_0), v_1(r_0)) = (0, 0)$. Therefore,

$$\vec{v}(r_0) := (v_0(r_0), v_1(r_0)) = (0, 0).$$

To make this statement about (u_0, u_1) we recall that (6.42) implies that

$$\|\pi_{r_0} \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq r_0)} = 0.$$

Lemma 6.5 then gives

$$\|\pi_{r_0}^\perp \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq r_0)} = 0,$$

and thus

$$\|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq r_0)} = 0,$$

which completes the proof since we know that $\lim_{r \rightarrow \infty} u_0(r) = 0$. \square

Proof of Lemma 6.15. Suppose that $\ell_0 = 0$. Then, by Claim 6.16, (u_0, u_1) must be compactly supported. Assume that $(u_0, u_1) \neq (0, 0)$. We can then define $\rho_0 > 0$ by

$$\rho_0 := \inf \left\{ \rho : \|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho)} = 0 \right\}$$

Let $\varepsilon > 0$ be a small number (to be determined below). Find $\rho_1 = \rho_1(\varepsilon)$ with $\frac{1}{2}\rho_0 < \rho_1 < \rho_0$ so that

$$0 < \|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 < \varepsilon \leq \delta_1^2$$

with δ_1 as in (6.43). We have

$$\begin{aligned} 3\rho_1^{-3} v_0^2(\rho_1) + \rho_1^{-1} v_1^2(\rho_1) + \int_{\rho_1}^{\infty} \left(\frac{1}{r} \partial_r v_0(r) \right)^2 dr + \int_{\rho_1}^{\infty} (\partial_r v_1(r))^2 dr &= \\ &= \|\pi_{\rho_1} \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 + \|\pi_{\rho_1}^\perp \vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 \\ &= \|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 < \varepsilon \end{aligned} \quad (6.62)$$

Using Lemma 6.8 with $R = \rho_1$ gives

$$\left(\int_{\rho_1}^{\infty} \left[\left(\frac{1}{r} \partial_r v_0(r) \right)^2 + (\partial_r v_1(r))^2 \right] dr \right)^{\frac{1}{2}} \lesssim \rho_1^{-\frac{11}{2}} |v_0(\rho_1)|^3 + \rho_1^{-\frac{5}{2}} |v_1(\rho_1)|^3 \quad (6.63)$$

Since $v_0(\rho_0) = v_1(\rho_0) = 0$ we can argue as in Corollary 6.9 and Corollary 6.10 to obtain

$$\begin{aligned} |v_0(\rho_1)| &= |v_0(\rho_1) - v_0(\rho_0)| \leq C_1 \varepsilon (|v_0(\rho_1)| + |v_1(\rho_1)|) \\ |v_1(\rho_1)| &= |v_1(\rho_1) - v_1(\rho_0)| \leq C_2 \varepsilon (|v_0(\rho_1)| + |v_1(\rho_1)|) \end{aligned}$$

where above we used that $\frac{1}{2}\rho_0 < \rho_1 < \rho_0$ to find constants C_1, C_2 depending only on ρ_0 which is fixed, and the uniform constant in (6.63), but not on ε . Combining the above estimates we get

$$(|v_0(\rho_1)| + |v_1(\rho_1)|) \leq C_3 \varepsilon (|v_0(\rho_1)| + |v_1(\rho_1)|) \quad (6.64)$$

which implies that $|v_0(\rho_1)| = |v_1(\rho_1)| = 0$ after choosing $\varepsilon > 0$ small enough. By (6.63) and (6.62) we see that

$$\|\vec{u}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)} = 0.$$

But this contradicts the definition of ρ_0 because $\rho_1 < \rho_0$. \square

We have finished the proof of Proposition 3.4 in the case where $\ell_0 = 0$. We next move on to the case $\ell_0 \neq 0$.

Case 2: $\ell_0 \neq 0$ cannot happen: To complete the proof of Proposition 3.4 we prove that $\ell_0 \neq 0$ is impossible. Indeed, we show that if $\ell_0 \neq 0$ then the solution $\vec{u}(t)$ to (1.2) (resp. (1.18), resp. (1.22)) with the compactness property as in Proposition 3.4 must be equal to a nonzero stationary solution, φ_{ℓ_0} , of (6.16) (resp. (6.6), resp. (6.7)).

On the other hand, we know by Lemma 6.3 (resp. Lemma 6.2) that $(\varphi_{\ell_0}, 0) \notin \dot{H}^{\frac{3}{2}}(\mathbb{R}^5)$ for any $\ell_0 \neq 0$, which gives us contradiction since by construction our solution $\vec{u}(t) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$.

The basic idea is to linearize about the elliptic solution φ_{ℓ_0} given by Lemma 6.2 (resp. Lemma 6.2) with the same leading order spacial asymptotics as the critical data $\vec{u}(0)$. In Steps 1, 2 we proved that

$$r^3 u_0(r) = \ell_0 + O(r^{-4}) \quad \text{as } r \rightarrow \infty.$$

Now, let $\varphi_{\ell_0}(r)$ be the solution from either Lemma 6.3 or Lemma 6.2 depending of course on which equation $\vec{u}(0)$ is initial data for. Hence φ_{ℓ_0} solves the relevant elliptic equation and satisfies

$$\varphi_{\ell_0}(r) = \ell_0 + O(r^{-4}) \quad \text{as } r \rightarrow \infty.$$

Next define $\vec{w}_{\ell_0}(0) = (w_{\ell_0,0}, w_{\ell_0,1})$ by

$$\begin{aligned} w_{\ell_0,0}(r) &:= u_0(r) - \varphi_{\ell_0}(r) \\ w_{\ell_0,1}(r) &:= u_1(r), \end{aligned} \tag{6.65}$$

and for all $t \in I_{\max}(\vec{u})$ set

$$w_{\ell_0}(t, r) := u(t, r) - \varphi_{\ell_0}(r). \tag{6.66}$$

We record various properties of $\vec{w}_{\ell_0} = (w_{\ell_0}, \partial_t w_{\ell_0})$. First, we have

$$\begin{aligned} v_{\ell_0,0}(r) &:= r^3 w_{\ell_0,0}(r) = O(r^{-4}) \quad \text{as } r \rightarrow \infty \\ v_{\ell_0,1}(r) &:= r \int_r^\infty w_{\ell_0,1}(\rho) \rho, d\rho = O(r^{-2}) \quad \text{as } r \rightarrow \infty \end{aligned} \tag{6.67}$$

Next, we write down the equation for \vec{w}_{ℓ_0} . If $\vec{u}(t)$ solves (1.2) and φ_{ℓ_0} solves (6.16) then $\vec{w}_{\ell_0}(t)$ solves

$$\begin{aligned} \partial_{tt} w_{\ell_0} - \partial_{rr} w_{\ell_0} - \frac{4}{r} \partial_r w_{\ell_0} &= 3\varphi_{\ell_0}^2 w_{\ell_0} + 3\varphi_{\ell_0} w_{\ell_0}^2 + w_{\ell_0}^3 \\ &=: \mathcal{N}_{\text{cubic}}(\varphi_{\ell_0}, w_{\ell_0}) \end{aligned} \tag{6.68}$$

If $\vec{u}(t)$ solves (1.18) or (1.22) then $\vec{w}_{\ell_0}(t)$ solves an equation of the form

$$\begin{aligned} \partial_{tt}w_{\ell_0} - \partial_{rr}w_{\ell_0} - \frac{4}{r}\partial_rw_{\ell_0} &= \mathcal{N}_{\text{w.m.}}(r, \varphi_{\ell_0}, w_{\ell_0}) \\ \mathcal{N}_{\text{w.m.}}(r, \varphi_{\ell_0}, w_{\ell_0}) &:= -2\frac{\mathcal{C}(2r\varphi_{\ell_0})-1}{r^2}w_{\ell_0} - \mathcal{S}(2r\varphi_{\ell_0})\frac{\mathcal{C}(2rw_{\ell_0})-1}{r^3} \\ &\quad - \mathcal{C}(2r\varphi_{\ell_0})\frac{(\mathcal{S}(2rw_{\ell_0})-2rw_{\ell_0})}{r^3} \end{aligned} \quad (6.69)$$

where if $\vec{u}(t)$ solves the \mathbb{S}^3 target equation then $\mathcal{C} = \cos$, $\mathcal{S} = \sin$ and if $\vec{u}(t)$ solves the \mathbb{H}^3 target equation we have $\mathcal{C} = \cosh$, $\mathcal{S} = \sinh$. In either case, we have the estimates

$$|\mathcal{N}_{\text{w.m.}}(r, \varphi_{\ell_0}, w_{\ell_0})| \lesssim |\varphi_{\ell_0}|^2 |w_{\ell_0}| + |\varphi_{\ell_0}| |w_{\ell_0}|^2 + |w_{\ell_0}|^3 \quad (6.70)$$

where again we have used our L^∞ control over ru and $r\varphi_\ell$ in the case of the \mathbb{H}^3 target equation.

The crucial point here is that by construction \vec{w}_{ℓ_0} inherits the main conclusions from Lemma 6.1 because φ_{ℓ_0} is stationary, i.e.,

$$\|\vec{w}_{\ell_0}(t)\|_{\dot{H}^1 \times L^2(r \geq R+|t|)} \rightarrow 0 \quad \text{as } |t| \rightarrow \infty \quad (6.71)$$

We are now in position to prove, much as in the case $\ell_0 = 0$, that we must have $\vec{w}_{\ell_0} \equiv (0, 0)$. We state this conclusion as a lemma.

Lemma 6.17. *Suppose $\ell_0 \neq 0$. Let \vec{w}_{ℓ_0} be as in (6.65), (6.66). Then, $\vec{w}_{\ell_0} \equiv (0, 0)$, that is, $\vec{u}(0) = (\varphi_{\ell_0}, 0)$ where φ_{ℓ_0} is given by either Lemma 6.3 or Lemma 6.2 (depending of course on which equation $\vec{u}(t)$ solves). This means that $\vec{u}(0) \notin \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ which is a contradiction.*

The proof of Lemma 6.17 will be very similar to the argument that was presented in the previous steps which completed the case $\ell_0 = 0$. We will thus omit many details here.

We have established the asymptotic behavior of \vec{w}_{ℓ_0} given in (6.67). However we recall that one of the keys to the argument in the previous steps was the estimate (6.28), as this gave a quantitative restriction on the proximity of $\vec{u}(0)$ to the plane $P(R)$. Here we establish a similar estimate for \vec{w}_{ℓ_0} . As before modify the Cauchy problems (6.68) and (6.69) on the interior of the cone $\mathcal{C}_R(t, r) := \{r \leq R + |t|\}$ for large R . As in the relevant sections in [25, 45, 20] we alter the right-hand-side of (6.68) and (6.69).

For each $R > 0$ we define $\varphi_{\ell_0, R}$ by

$$\varphi_{\ell_0, R}(t, r) := \begin{cases} \varphi_{\ell_0}(R + |t|) & \text{for } 0 \leq r \leq R + |t| \\ \varphi_{\ell_0}(r) & \text{for } r \geq R + |t| \end{cases} \quad (6.72)$$

where φ_{ℓ_0} is the relevant elliptic solution depending on whether we are dealing with solutions to (1.2), (1.18), or (1.22). Next, set

$$\begin{aligned} \mathcal{N}_{R, \text{cubic}}(t, r, w_{\ell_0}) &:= \mathcal{N}_{\text{cubic}}(\varphi_{\ell_0, R}, w_{\ell_0}) \\ \mathcal{N}_{R, \text{w.m.}}(t, r, w_{\ell_0}) &:= \begin{cases} \mathcal{N}_{\text{w.m.}}(R + |t|, \varphi_{\ell_0, R}, w_{\ell_0}) & \text{for } 0 \leq r \leq R + |t| \\ \mathcal{N}_{\text{w.m.}}(r, \varphi_{\ell_0, R}, w_{\ell_0}) & \text{for } r \geq R + |t| \end{cases} \end{aligned}$$

Using (6.17) and (6.8) together with (6.70), we can deduce that for R large enough we have

$$|\mathcal{N}_{R,\text{cubic}}(t, r, w_{\ell_0})| \lesssim \begin{cases} (R + |t|)^{-6} |w_{\ell_0}| + (R + |t|)^{-3} |w_{\ell_0}|^2 + |w_{\ell_0}|^3, \\ \text{for } 0 \leq r \leq R + |t| \end{cases} \quad (6.73)$$

$$|\mathcal{N}_{R,\text{cubic}}(t, r, w_{\ell_0})| \lesssim \begin{cases} (R + |t|)^{-6} |w_{\ell_0}| + (R + |t|)^{-3} |w_{\ell_0}|^2 + |w_{\ell_0}|^3, \\ \text{for } 0 \leq r \leq R + |t| \end{cases} \quad (6.74)$$

Because we have the same estimates for $\mathcal{N}_{R,\text{cubic}}$ and $\mathcal{N}_{R,\text{w.m.}}$ above we will simply write \mathcal{N}_R for both of them in what follows. We consider a modified Cauchy problem. As in the set-up for Lemma 6.6 we define a smooth radial function $\chi \in C^\infty(\mathbb{R}^5)$ with $\chi(r) = 1$ for $r \geq 1$ and $\chi(r) = 0$ on $r \leq 1/2$. Rescale to define $\chi_R(r) := \chi(r/R)$ and for each $R > 0$ we consider:

$$\begin{aligned} w_{tt} - w_{rr} - \frac{4}{r} w_r &= \tilde{\mathcal{N}}_R(t, r, w) \\ \tilde{\mathcal{N}}_R(t, r, w) &:= \chi_R \mathcal{N}_R(t, r, w) \\ \vec{w}(0) &= (w_0, w_1) \in \dot{H}^1 \times L^2(\mathbb{R}^5) \end{aligned} \quad (6.75)$$

The idea here is that we have introduced spacial decay as well as time integrability into the potential terms that will then allow these to be treated in a perturbative manner. We have also introduced the cut-off χ_R . As in the Lemma 6.6 this removes the super-critical nature of the nonlinearity and allows us to treat the right-hand-side perturbatively in the *energy space*. This is an analog of Lemma 6.6 but here we have linearized about a nontrivial elliptic solution φ_{ℓ_0} . As before we define the norm $Z(I)$ by

$$\|\vec{w}\|_{Z(I)} = \|w\|_{L_t^2(I; L_x^5(\mathbb{R}^5))} + \|\vec{w}(t)\|_{L_t^\infty(I; \dot{H}^1 \times L^2)}$$

Lemma 6.18. *There exists $R_2 > 0$ and $\delta_2 > 0$ small enough such that for every $R > R_2$ and all data $\vec{w}(0) = (w_0, w_1) \in \dot{H}^1 \times L^2(\mathbb{R}^5)$ with*

$$\|\vec{w}(0)\|_{\dot{H}^1 \times L^2(\mathbb{R}^5)} < \delta_2$$

there exists a unique global solution $\vec{w}(t) \in \dot{H}^1 \times L^2$ to (6.75). Moreover, $\vec{w}(t)$ satisfies

$$\|w\|_{Z(\mathbb{R})} \lesssim \|\vec{w}(0)\|_{\dot{H}^1 \times L^2(\mathbb{R}^5)} \lesssim \delta_2 \quad (6.76)$$

Finally, if we denote the free evolution with the same data by $w_L(t) := S(t)\vec{h}(0) \in \dot{H}^1 \times L^2(\mathbb{R}^5)$, then we have

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|\vec{w}(t) - \vec{w}_L(t)\|_{\dot{H}^1 \times L^2} &\lesssim R^{-4} \|\vec{w}(0)\|_{\dot{H}^1 \times L^2} + R^{-5/2} \|\vec{w}(0)\|_{\dot{H}^1 \times L^2}^2 \\ &\quad + R^{-1} \|\vec{w}(0)\|_{\dot{H}^1 \times L^2}^3 \end{aligned} \quad (6.77)$$

Proof. The proof follows from (6.73) and (6.74) and is very similar to the proof of Lemma 6.6. In particular by a standard argument it suffices to control $\tilde{\mathcal{N}}_R$ in $L_t^1 L_x^2$ using the estimates (6.73) and (6.74). We omit the details. See for example [45, Lemma 5.17] for a more detailed argument of a similar result. \square

With Lemma 6.18 in hand, we argue exactly as in the proof of Lemma 6.5 to prove the key inequality:

Lemma 6.19. *There exists $R_2 > 0$ such that for all $R > R_2$ and for all $t \in I_{\max}(u)$ we have*

$$\begin{aligned} \|\pi_R^{\frac{1}{2}} \vec{w}_{\ell_0}(t)\|_{\dot{H}^1 \times L^2(r \geq R)} &\lesssim R^{-4} \|\pi_R \vec{w}_{\ell_0}(t)\|_{\dot{H}^1 \times L^2(r \geq R)} + R^{-\frac{5}{2}} \|\pi_R \vec{w}_{\ell_0}(t)\|_{\dot{H}^1 \times L^2(r \geq R)}^2 \\ &\quad + R^{-1} \|\pi_R \vec{w}_{\ell_0}(t)\|_{\dot{H}^1 \times L^2(r \geq R)}^3 \end{aligned}$$

where $\pi_R, \pi_R^{\frac{1}{2}}$ are as in Proposition 6.4. We note that the constant above is uniform in $t \in I_{\max}(u)$.

Next, we prove that $(\partial_r w_{\ell_0,0}, w_{\ell_0,1})$ must be compactly supported.

Claim 6.20. *Let \vec{w}_{ℓ_0} be as in (6.65). Then $(\partial_r w_{\ell_0,0}, w_{\ell_0,1})$ must be compactly supported.*

Proof of Claim 6.20. To prove the claim, we pass to the \vec{v}_{ℓ_0} formulation. With $(v_{\ell_0,0}, v_{\ell_0,1})$ defined as in (6.67) we can conclude that for all $R > R_2$ large enough we have

$$\begin{aligned} \int_R^\infty \left(\frac{1}{r} \partial_r v_{\ell_0,0}(r) \right)^2 dr + \int_R^\infty (\partial_r v_{\ell_0,1}(r))^2 dr &\lesssim R^{-19} v_{\ell_0,0}^2(R) + R^{-11} v_{\ell_0,0}^4(R) \\ &\quad + R^{-11} v_{\ell_0,0}^6(R) + R^{-17} v_{\ell_0,1}^2(R) + R^{-7} v_{\ell_0,1}^4(R) + R^{-5} v_{\ell_0,1}^6(R) \\ &\lesssim R^{-11} (v_{\ell_0,0}^2(R) + v_{\ell_0,1}^2(R)) \end{aligned} \tag{6.78}$$

where the first inequality above follows by rephrasing Lemma 6.19 in terms of $\vec{v}_{\ell_0} = (v_{\ell_0,0}, v_{\ell_0,1})$ by using (6.42), and the last line following from the decay estimates in (6.67).

Next, arguing as in Corollary 6.9, we can prove difference estimates, i.e, for all $R_2 \leq r \leq r' \leq 2r$ we have

$$\begin{aligned} |v_{\ell_0,0}(r) - v_{\ell_0,0}(r')| &\lesssim r^{-4} (v_{\ell_0,0}^2(r) + v_{\ell_0,1}^2(r))^{\frac{1}{2}}, \\ |v_{\ell_0,1}(r) - v_{\ell_0,1}(r')| &\lesssim r^{-5} (v_{\ell_0,0}^2(r) + v_{\ell_0,1}^2(r))^{\frac{1}{2}}. \end{aligned} \tag{6.79}$$

Defining the vector $\vec{v}_{\ell_0} = (v_{\ell_0,0}, v_{\ell_0,1})$, where

$$|\vec{v}_{\ell_0}(r)|^2 := v_{\ell_0,0}^2(r) + v_{\ell_0,1}^2(r)$$

this means that

$$|\vec{v}_{\ell_0}(r) - \vec{v}_{\ell_0}(r')| \lesssim r^{-4} |\vec{v}_{\ell_0}(r)|.$$

We can then prove that for fixed $r_0 \geq R_2$ large enough,

$$|\vec{v}_{\ell_0}(2^{n+1}r_0)| \geq \frac{3}{4} |\vec{v}_{\ell_0}(2^n r_0)|.$$

Iterating this, we see that for each $n \in \mathbb{N}$, we have

$$|\vec{v}_{\ell_0}(2^n r_0)| \geq \left(\frac{3}{4} \right)^n |\vec{v}_{\ell_0}(r_0)|.$$

On the other hand, using (6.67) we have

$$|\vec{v}_{\ell_0}(2^n r_0)| \lesssim (2^n r_0)^{-2}.$$

Combining the last two lines we obtain

$$3^n |\vec{v}_{\ell_0}(r_0)| \lesssim 1,$$

which then implies that $\vec{v}_{\ell_0}(r_0) = (0, 0)$. Plugging this last fact into (6.78) yields

$$\int_{r_0}^{\infty} \left(\frac{1}{r} \partial_r v_{\ell_0,0}(r) \right)^2 dr + \int_{r_0}^{\infty} (\partial_r v_{\ell_0,1}(r))^2 dr = 0.$$

Therefore,

$$\begin{aligned} & \|\vec{u}_{\ell_0}\|_{\dot{H}^1 \times L^2(r \geq r_0)}^2 = \\ &= \int_{r_0}^{\infty} \left(\frac{1}{r} \partial_r v_{\ell_0,0}(r) \right)^2 dr + \int_{r_0}^{\infty} (\partial_r v_{\ell_0,1}(r))^2 dr + 3r_0^{-3} v_{\ell_0,0}^2(r_0) + r_0^{-1} v_{\ell_0,1}^2(r_0) = 0 \end{aligned}$$

which implies that $(\partial_r u_{\ell_0,0}, u_{\ell_0,1})$ must be compactly supported. \square

We can now complete the proof of Lemma 6.17 by showing that in fact, $\vec{w}_{\ell_0} \equiv (0, 0)$.

Proof of Lemma 6.17. The proof follows the same argument as the proof of Lemma 6.15. Suppose that

$$(\partial_r w_{\ell_0,0}, w_{\ell_0,1}) \neq (0, 0).$$

By Claim 6.20, $(\partial_r w_{\ell_0,0}, w_{\ell_0,1})$ is compactly supported. Then we can define $\rho_0 > 0$ by

$$\rho_0 := \inf \left\{ \rho > 0 : \|\vec{w}_{\ell_0}\|_{\dot{H}^1 \times L^2(r \geq \rho)} = 0 \right\}$$

Let $\varepsilon > 0$ be a small number (to be determined below). Let $\rho_1 = \rho_1(\varepsilon)$ be chosen so that

$$\frac{1}{2}\rho_0 < \rho_1 < \rho_0, \quad \text{and} \quad \rho_0 - \rho_1 < \varepsilon, \quad (6.80)$$

$$0 < \|\vec{u}_{\ell_0}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 < \delta_2^2 \quad (6.81)$$

with δ_2 is as in Lemma 6.18. With $(v_{\ell_0,0}, v_{\ell_0,1})$ as above,

$$\begin{aligned} & 3\rho_1^{-3} v_{\ell_0,0}^2(\rho_1) + \rho_1^{-1} v_{\ell_0,1}^2(\rho_1) + \int_{\rho_1}^{\infty} \left(\frac{1}{r} \partial_r v_{\ell_0,0}(r) \right)^2 dr + \int_{\rho_1}^{\infty} (\partial_r v_{\ell_0,1}(r))^2 dr = \\ &= \|\pi_{\rho_1} \vec{u}_{\ell_0}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 + \|\pi_{\rho_1}^{\perp} \vec{u}_{\ell_0}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 \\ &= \|\vec{u}_{\ell_0}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)}^2 < \delta_2^2 \end{aligned} \quad (6.82)$$

Setting $R = \rho_1$ in (6.78) then yields

$$\begin{aligned} & \left(\int_{\rho_1}^{\infty} \left[\left(\frac{1}{r} \partial_r v_{\ell_0,0}(r) \right)^2 + (\partial_r v_{\ell_0,1}(r))^2 \right] dr \right)^{\frac{1}{2}} \lesssim \rho_1^{-\frac{11}{2}} \left(|v_{\ell_0,0}(\rho_1)|^2 + |v_{\ell_0,1}(\rho_1)|^2 \right)^{\frac{1}{2}} \\ & \lesssim \rho_0^{-\frac{11}{2}} \left(|v_{\ell_0,0}(\rho_1)|^2 + |v_{\ell_0,1}(\rho_1)|^2 \right)^{\frac{1}{2}} \end{aligned} \quad (6.83)$$

where the assumption $\frac{1}{2}\rho_0 < \rho_1 < \rho_0$ enters in the last line above. Because $v_{\ell_0,0}(\rho_0) = v_{\ell_0,1}(\rho_0) = 0$ we can argue exactly as in Corollary 6.9 to obtain

$$\begin{aligned} |v_{\ell_0,0}(\rho_1)|^2 &= |v_{\ell_0,0}(\rho_1) - v_{\ell_0,0}(\rho_0)|^2 \leq C_1 \varepsilon^3 (|v_{\ell_0,0}(\rho_1)|^2 + |v_{\ell_0,1}(\rho_1)|^2) \\ |v_{\ell_0,1}(\rho_1)|^2 &= |v_{\ell_0,1}(\rho_1) - v_{\ell_0,1}(\rho_0)|^2 \leq C_2 \varepsilon (|v_{\ell_0,0}(\rho_1)|^2 + |v_{\ell_0,1}(\rho_1)|^2) \end{aligned}$$

with C_1, C_2 depending only ρ_0 which is fixed, and the uniform constant in (6.83), but not on ε . Combining above gives

$$(|v_{\ell_0,0}(\rho_1)|^2 + |v_{\ell_0,1}(\rho_1)|^2) \leq C_3 \varepsilon (|v_{\ell_0,0}(\rho_1)|^2 + |v_{\ell_0,1}(\rho_1)|^2), \quad (6.84)$$

This shows that $|v_{\ell_0,0}(\rho_1)| = |v_{\ell_0,1}(\rho_1)| = 0$ once $\varepsilon > 0$ is chosen small enough. By (6.83) and the equalities in (6.82) we see that

$$\|\vec{w}_{\ell_0}(0)\|_{\dot{H}^1 \times L^2(r \geq \rho_1)} = 0.$$

But this contradicts the definition of ρ_0 since $\rho_1 < \rho_0$. Thus, $(\partial_r w_{\ell_0,0}, w_{\ell_0,1}) \equiv (0, 0)$. Since $w_{\ell_0}(r) \rightarrow 0$ as $r \rightarrow \infty$ we also deduce that $(w_{\ell_0,0}, w_{\ell_0,1}) \equiv (0, 0)$. \square

To clarify this lengthy argument we summarize the proof of Proposition 3.4.

Proof of Proposition 3.4. Let $\vec{u}(t)$ be a solution to (1.2) (resp. (1.18), resp. (1.22)) with the compactness property as in Proposition 3.4. By Lemma 6.7 there exists $\ell_0 \in \mathbb{R}$ so that

$$\begin{aligned} |r^3 u_0(r) - \ell_0| &= O(r^{-4}) \quad \text{as } r \rightarrow \infty \\ \left| r \int_r^\infty u_1(\rho) \rho d\rho \right| &= O(r^{-2}) \quad \text{as } r \rightarrow \infty \end{aligned} \quad (6.85)$$

If $\ell_0 \neq 0$ then by Lemma 6.17, $\vec{u}(0) = (u_0, u_1) = (\varphi_{\ell_0}, 0)$ where φ_{ℓ_0} is a nonzero solution to (6.16) (resp. (6.6), resp. (6.7)) given by Lemma 6.3 (resp. Lemma 6.2). However, this is impossible since $\varphi_{\ell_0} \notin \dot{H}^{\frac{3}{2}}(\mathbb{R}^5)$, while on the other hand we know that $\vec{u}(0) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$.

Thus, we must have $\ell_0 = 0$. By Lemma 6.15 we can conclude that $\vec{u}(0) = (0, 0)$, proving Proposition 3.4. \square

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